SOLUTION SKETCHES. You are expected to fill in the details

Integration Techniques, Week 2, January 24, 2018

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1. Construct rules for decomposing rational functions of the following forms. Note that in each case, we must have the degree of the numerator strictly less than the degree of the denominator.

- $\dfrac{p(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \dfrac{A_1}{a_1x+b_1} + \dfrac{A_2}{a_2x+b_2} + \cdots + \dfrac{A_n}{a_nx+b_n}$

- $\dfrac{p(x)}{(ax+b)^n} = \dfrac{A_1}{(ax+b)} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_n}{(ax+b)^n}$

- If $q_i(x)$ are all irreducible quadratic polynomials $\dfrac{p(x)}{q1(x)q_2(x)\cdots q_n(x)} = \dfrac{A_1x+B_1}{q_1(x)} + \dfrac{A_2x+B_2}{q_2(x)} + \cdots + \dfrac{A_nx+B_n}{q_n(x)}$

- If $q(x)$ is an irreducible quadratic polynomial $\dfrac{p(x)}{(q(x))^n} = \dfrac{A_1x+B_1}{q(x)} + \dfrac{A_2x+B_2}{(q(x))^2} + \cdots + \dfrac{A_nx+B_n}{(q(x))^n}$

2. Evaluate $\displaystyle\int_3^4 \frac{2x^2+4x-7}{x^2+x-6}\,dx$ SEE CLASS NOTES. Use long division + partial fraction decomposition

3. Evaluate the following integrals:

(a) $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$ Set $x = 4\sin\theta$, $dx = 4\cos\theta\, d\theta$ $\rightsquigarrow \displaystyle\int_0^{\pi/3} 4^3 \sin^3\theta\, d\theta$

(b) $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt$ Set $x = \sec\theta$, $dx = \tan\theta\sec\theta\, d\theta$ $\rightsquigarrow \displaystyle\int_{\pi/4}^{\pi/3} \cos^2\theta\, d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2}(1+\cos(2\theta))\,d\theta$

(c) $\displaystyle\int_0^2 x^3\sqrt{x^2+4}\,dx$ Set $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$ $\rightsquigarrow \displaystyle\int_0^{\pi/4} 32\tan^3\theta\sec^3\theta\, d\theta$ Set $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$

(d) $\displaystyle\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx$ Set $x = \frac{2}{3}\sin\theta$, $dx = \frac{2}{3}\cos\theta\, d\theta$ $\rightsquigarrow \displaystyle\int_0^{\pi/2} \frac{32}{3^4}\sin^3\theta\cos^2\theta\, d\theta$

(e) $\displaystyle\int_0^{\pi/2} \frac{\cos(t)}{\sqrt{1+\sin^2(t)}}\,dt$ Set $u = \sin t$. $du = \cos(t)\, dt$ $\rightsquigarrow \displaystyle\int_0^1 \frac{1}{\sqrt{1+u^2}}\,du$ Set $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$

4. Come up with a set of rules for evaluating integrals containing the following, where a is a constant:

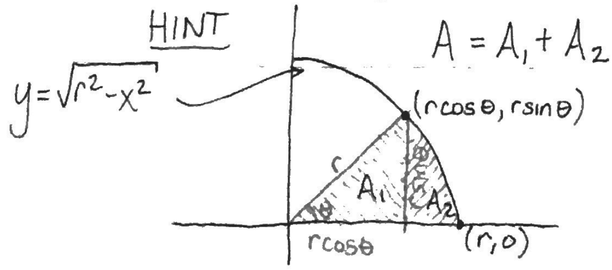
(a) $\sqrt{a^2-x^2}$ Set $x = a\sin\theta$, $\theta \in [-\pi/2, \pi/2]$

(b) $\sqrt{a^2+x^2}$ Set $x = a\tan\theta$ $\theta \in (-\pi/2, \pi/2)$

(c) $\sqrt{x^2-a^2}$ Set $x = a\sec\theta$ $\theta \in [0, \pi/2)$ or $[\pi, \frac{3\pi}{2})$

5. Prove the formula $A = \frac{1}{2}r^2\theta$ for the area of a sector of a ciricle with radius r and central angle θ.

HINT



$A = A_1 + A_2 = \frac{1}{2} r\sin\theta\, r\cos\theta + A_2$

how can you compute this area?

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