OUTLINE OF SOLUTIONS

Integration by Substitution, January 16, 2018

1. Develop a list of functions for which you immediately know the antiderivative of.
   - $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$
   - $\int x^{-1}\,dx = \log|x| + c$
   - $\int e^x\,dx = e^x + c$
   - $\int \frac{1}{1+x^2}\,dx = \arctan(x) + c$
   - $\int \sin(x)\,dx = -\cos(x) + c$, $\int \cos(x)\,dx = \sin(x) + c$, $\int \sec^2(x)\,dx = \tan(x) + c$

2. Evaluate $\int_0^{\pi/2} \cos(x)\sin(x)\,dx$

   Set $u = \sin(x)$, $du = \cos(x)\,dx$. $u: 0 \to 1$ as $x: 0 \to \pi/2$

   So, $\int_0^{\pi/2} \cos(x)\sin(x)\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

3. Evaluate $\int_1^2 \frac{2t}{t^2+5}\,dt$ Set $u = t^2+5$, $du = 2t\,dt$. As $t: 1 \to 2$, $u: 6 \to 9$

   $\int_1^2 \frac{2t}{t^2+5}\,dt = \int_6^9 \frac{du}{u} = \log|u|\Big|_6^9 = \log\left(\frac{9}{6}\right) = \log\left(\frac{3}{2}\right)$

4. Determine the appropriate u-substitution for the following integrals:

   (a) $\int_0^2 x^2 \sin(x^3+1)\,dx$ $\quad u = x^3+1$, $du = 3x^2\,dx$, $x^2\,dx = \frac{du}{3}$ $\rightsquigarrow \int_1^9 \sin(u)\,du$

   (b) $\int_1^e \frac{\sqrt{\log(x)}}{x}\,dx$ $\quad u = \log(x)$ $du = \frac{1}{x}\,dx$ $\rightsquigarrow \int_0^1 \sqrt{u}\,du$

   (c) $\int_3^5 e^{4x+10}\,dx$ $\quad u = 4x+10$, $du = 4\,dx$, $dx = \frac{du}{4}$ $\rightsquigarrow \int_{22}^{30} \frac{1}{4} e^u\,du$

   (d) $\int_0^1 \frac{x^2+1}{\sqrt{x+1}}\,dx$ $\quad u = \sqrt{x+1}$, $u^2 - 1 = x$, $dx = 2u\,du$ $\rightsquigarrow \int_1^{\sqrt{2}} \frac{(u^2-1)^2+1}{u}\,du$

5. Evaluate the following integrals

   (a) $\int_1^2 \frac{2x}{x^4+1}\,dx$ $\quad u = x^2$, $du = 2x\,dx$ $\rightsquigarrow \int_1^4 \frac{1}{u^2+1}\,du = \arctan(u)\Big|_1^4$

   (b) $\int_0^{\pi/2} \sin^3(x)\cos(x)\,dx$ $\quad u = \sin(x)$ $du = \cos(x)\,dx$ $\rightsquigarrow \int_0^1 u^3\,du = \frac{u^4}{4}\Big|_0^1 = \frac{1}{4}$

   (c) $\int_0^{\pi/2} \sin^3(x)\,dx = \int_0^{\pi/2} \sin(x)(1-\cos^2(x))\,dx$ Set $u = \cos(x)$, $du = -\sin(x)\,dx$

   (d) $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$

   Set $u = x^2+1$ $\quad du = 2x\,dx$

   $x = \sqrt{u-1}$ $\quad dx = \frac{du}{2\sqrt{u-1}}$

   $\rightsquigarrow \int_0^2 \frac{(\sqrt{u-1})^3}{\sqrt{u}} \frac{du}{2\sqrt{u-1}} = \int_0^2 \frac{(\sqrt{u-1})^2}{2\sqrt{u}}\,du$

   $= \frac{2^{3/2}}{3} - 2^{1/2}$