Work, Part 2, February 9, 2018

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1. Suppose one end of a 100 m steel rope with a mass of 91 kg is attached to the lip of a roof of a 120 m tall building. Calculate the work done to hoist the rope up to the roof.

   SOLUTION: The relevant force in this problem is the force due to gravity, so we have $F(x) = m(x)g$, where $m(x)$ is the mass of the rope at a particular value of x and g is the acceleration due to gravity. Let x be the distance from the roof, to a particular point on the rope. Since the rope has a mass of 91 kg and is 100 m long, we have that for a small increment of the rope at x - call it Δx - the mass of that increment is $\frac{91}{100}\Delta x$. Hence, $F(x) = \frac{91}{100}\Delta x g$. Thus, the work done to raise the rope the full 100 m is:

   $$W = \int_0^{100} \frac{91}{100} gx \, dx$$

2. Suppose one end of a 100 m steel rope of mass 91 kg is attached to the roof a 40 m tall building. Calculate the work done to hoist the rope up to the roof.

   SOLUTION: This problem is similar to the last, in that we still have 100 m of rope and we need to consider the force due to gravity. However, we have 60 m of the rope lying on the ground. Let x denote the amount of rope that has been lifted onto the roof. When we begin to pull the rope, for every meter we pull up, we pull a meter out of the pile on the ground. This continues until the pile of rope on the ground is no longer on the ground; that is, after we pull up 60 m of the rope with a mass of 40(.91) kg. Thus, for $0 \leq x \leq 60$, $F(x) = m(x)g = (0.91)40g$.
   After the first 60 meters of rope have been brought to the roof, there is now 40 meters of rope dangling with nothing left coiled below. Therefore, as we bring up these last 40 m, there is less and less rope hanging, and so the mass of the rope (and hence the force we exert) is decreasing. It decreases linearly, since the rope has constant density. Each meter of rope we bring up decreases the mass by .91 kg, and so $F(x) = m(x)g = (91 - .91x)g$, where $60 \leq x \leq 100$.
   Finally, we have that the total work done to hoist up the rope will be:

   $$W = \int_0^{60} (.91)40g \, dx + \int_{60}^{100} (100 - .91x)g \, dx$$

   Alternatively, recognize that for the first 60 meters of rope, the force is constant. You are always holding 40 m of rope, with the force being $F(x) = m(x)g = (0.91)40g$. Then, the remaining 40 meters of rope is the same system as in question 1, above. So, the work required for those 40 meters can be represented by $W = \int_0^{40} \frac{91}{100} gx \, dx$. Then, the total work will be $W = (0.91)40g + \int_0^{40} \frac{91}{100} gx \, dx$

3. We have a cable that has a mass of 2 kg/m attached to a bucket filled with coal that weighs 400 kg. The bucket is at the bottom of a 500 m mine shaft. Determine the work done to lift the bucket from the midpoint to the top.

   SOLUTION: In this problem, we once again have the force due to gravity, $F = mg$. This job will get easier as we pull the cable up the shaft. Indeed, after we pull x meters of the cable up to the top, the mass that is hanging down is $m(x) = \textit{mass of bucket + mass of chain after pulling up x meters} = 400 + (1000 - 2x) = 1400 - 2x$. Hence, the force exerted is $F(x) = m(x)g = (1400 - 2x)g$. Finally, the work done to hoist the bucket up from the midpoint, which means we are pulling 250 meters of rope is:

   $$W = \int_{250}^{500} (1400 - 2x)g \, dx$$

4. A bucket with a mass of 30 kg when filled with sand needs to be lifted to the top of a 20 meter tall building. We have a rope that has a mass of 0.2 kg/m that takes 1 meter to secure to the bucket. Once the bucket reaches the top of the building, it has a mass of only 19 kg because there is a hole in the bottom and sand was leaking out at a constant rate while it was being lifted to the top of the building. Find the work done lifting the bucket, sand and rope to the top of the building.

   SOLUTION: Again, we let x be the amount of rope that we have pulled up to the top of the building. Similar to all the other problems, the force we need to consider is $F = mg$. Here, we have multiple elements of the system contributing to mass. We have the mass of the bucket, which is changing since it is leaking, as well as the mass of the rope, which is getting lighter as we increase x. We know that after pulling up the rope for x meters, the mass - call it $m(x)$ - is $m(x) = \textit{mass of bucket after x meters} + \textit{mass of rope after x meters} = m_b(x) + m_r(x)$. Let's deal with each case separately.

   We know that the rope has an initial mass of 4.2 kg, since we have 21 meters of rope at 0.2 kg/m (remember, 1 meter is around the bucket!!!). So, $m_r(x) = 4.2 - 0.2x$. In regards to the mass of the bucket, we know that after 20 meters, the mass is 19 kg. Since the sand was leaking out at a constant rate, we have that the bucket is loosing 11 kg per 20m. So, $m_b(x) = 30 - \frac{11}{20}x$.

   So, we have that the total mass after pulling up x meters of rope is $m(x) = m_b(x) + m_r(x) = 30 - \frac{11}{20} + 4.2 - 0.2x = 34.2 - \frac{15}{20}x$. Hence, the total force after x meters have been pulled up is $F(x) = (34.2 - \frac{15}{20}x)g$. Finally, the work done is:

$$W = \int_0^{20} (34.2 - \frac{15}{20}x)g\,dx$$