QUIZ 6

First name: SOLUTIONS

Last name: See page 2 for comments on common mistakes I observed.

Student number:

Recitation section:

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1. If $p > \frac{1}{4}$, use the Integral Test to determine if $\sum_{n=1}^{\infty} \frac{\log(n)}{n^{4p}}$ converges or diverges.

If $\int_1^{\infty} \frac{\log(x)}{x^{4p}}\,dx$ converges, then $\sum_{n=1}^{\infty} \frac{\log(n)}{n^{4p}}$ converges

By definition: $\int_1^{\infty} \frac{\log(x)}{x^{4p}}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{\log(x)}{x^{4p}}\,dx$

Use integration by parts with $u = \log(x)$ $du = \frac{1}{x}dx$ $dv = \frac{1}{x^{4p}}dx$ $V = \frac{x^{-4p+1}}{-4p+1}$

So: $\lim_{b\to\infty} \int_1^{b} \frac{\log(x)}{x^{4p}}\,dx = \lim_{b\to\infty} \left[\frac{\log(x)\,x^{-4p+1}}{-4p+1}\Big|_1^b - \int_1^b \frac{x^{-4p+1}}{-4p+1}\cdot\frac{1}{x}\,dx \right]$

$= \lim_{b\to\infty} \left[\frac{\log(b)}{b^{4p-1}(4p+1)} - \int_1^b \frac{x^{-4p}}{-4p+1}\,dx \right] = \frac{1}{-4p+1} \lim_{b\to\infty} \left[\frac{\log(b)}{b^{4p-1}} - \left[\frac{x^{-4p+1}}{-4p+1}\right]_1^b \right]$

$= \frac{1}{-4p+1} \lim_{b\to\infty} \left[\frac{1}{b\,b^{4p}(4p-1)} - \frac{b^{-4p+1}}{-4p+1} + \frac{1}{-4p+1} \right]$ by l'hôpital's Rule since $4p-1>0$.

$= \frac{1}{(-4p+1)^2} \lim_{b\to\infty} \left[\frac{1}{b^{4p+1}} \xrightarrow{} 0 - \frac{1}{b^{4p-1}} \xrightarrow{} 0 + 1 \right] = \frac{1}{(-4p+1)^2}$ $\therefore$ the integral converges. $\Rightarrow$ the series converges.

2. Find the first four non-zero terms of the power series representation centred at $x = 0$ for the function $f(x) = \frac{x^2}{3+6x}$. What is the radius of convergence?

$\frac{x^2}{3+6x} = x^2 \cdot \frac{1}{3+6x} = \frac{x^2}{3}\left(\frac{1}{1-(-2x)}\right)$

$= \frac{x^2}{3}\left(\sum_{n\geq 0} (-2x)^n\right)$ for $|2x| < 1$

$= \frac{x^2}{3} \sum_{n\geq 0} (-2)^n x^n$, for $|x| < 1/2$

$= \frac{1}{3} \cdot \sum_{n\geq 0} (-1)^n 2^n x^{n+2}$

$= \frac{1}{3}\left(x^2 - 2x^3 + 4x^5 - 8x^6 + \dots\right)$ for $|x| < 1/2$

$= \frac{x^2}{3} - \frac{2x^3}{3} + \frac{4x^5}{3} - \frac{8x^6}{3} + \dots$ for $|x| < 1/2$ $\Rightarrow$ RADIUS OF CONVERGENCE $= \frac{1}{2}$

QUIZ 6

First name: Observed Common Mistakes

Last name:

Student number:

Recitation section:

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1. If $p > \frac{1}{3}$, use the Integral Test to determine if $\sum_{n=1}^{\infty} \frac{\log(n)}{n^{3p}}$ converges or diverges.

- In this problem, I saw three major errors.

① Remember, $\int_1^\infty f(x)\,dx = \lim_{b\to\infty}\int_1^b f(x)\,dx$ by definition. To write $f(x)\Big|_1^\infty$ without a limit is meaningless, because $f(\infty)$ is not well-defined.

② The integral test says that for $\sum_{n\geq 1} a_n$, if $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n\geq 1} f(n)$ converges.

It does NOT say that $\int_1^\infty f(x)\,dx = \sum_{n\geq 1} f(n)$

Think about why this is false. That is, construct a counterexample.

③ If you got to the point of having no integrals to compute, you would have been left with $\lim_{b\to\infty}\frac{\log(b)}{b^{4p-1}}$. To compute this limit, you need l'hôpital's since $4p-1>0$.

2. Find the first four non-zero terms of the power series representation centred at $x = 0$ for the function $f(x) = \frac{x^2}{2+6x}$. What is the radius of convergence?

Two major mistakes here:

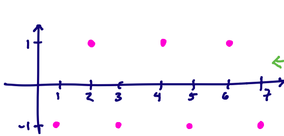
① READ THE QUESTION. It asked for the radius of convergence and first four terms.

② Many of you wanted to test the endpoints. Generally, this is important to do for power series, especially if using the ratio test to find the interval of convergence. However, since we know $\frac{1}{1-u} = \sum_{n\geq 0} u^n$ for $|u|<1$ we do not need to use the ratio test or test the endpoints and only make the substitution for $|u|<1$.

Beyond that, many claimed that the power series converged at the right endpoint. This is false. If you plug in the right endpoint, you get $\sum_{n\geq 1}(-1)^n = -1+1-1+\dots$ which DIVERGES. Yes, it may look like it sums to 0, but it diverges because

① the sequence of partial sums diverges

OR ② By the divergence test since $\lim_{n\to\infty}(-1)^n \neq 0$.



This sequence is not converging to anything.