QUIZ 7

First name: SOLUTIONS

Last name:

Student number:

Recitation section:

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1. Calculate the Taylor polynomial $T_4(x)$ centred at $x = \frac{\pi}{2}$ for $f(x) = \sin(x)$.

By definition, $T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(\pi/2)}{n!}(x-\pi/2)^n = f(\frac{\pi}{2}) + f'(\pi/2)(x-\pi/2) + \frac{f''(\pi/2)}{2}(x-\pi/2)^2 + \frac{f'''(\pi/2)}{3!}(x-\pi/2)^3 + \frac{f^{(4)}(\pi/2)}{4!}(x-\pi/2)^4$

So, we need $f(\pi/2), f'(\pi/2), f''(\frac{\pi}{2})$, and $f^{(4)}(\pi/2)$ for $f(x) = \sin(x)$

We make a table to condense our work:

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{2})$
0	$\sin(x)$	$\sin(\pi/2) = 1$
1	$\cos(x)$	$\cos(\pi/2) = 0$
2	$-\sin(x)$	-1
3	$-\cos(x)$	0
4	$\sin(x)$	1

This isn't a necessary note to solve this problem, but notice that we can see that $f^{(2n)}(\pi/2)$ follows the pattern $(-1)^n$.

Hence, $T_4(x) = 1 + 0 - \frac{1}{2}(x-\pi/2)^2 + 0 + \frac{1}{4!}(x-\pi/2)^4 = \sum_{n=0}^{4} \frac{(-1)^n}{(2n)!}(x-\pi/2)^{2n}$

2. What is the minimal degree Taylor polynomial about $x = 0$ that you need to calculate $\cos(0.5)$ to strictly within $(2^{-8})/8!$ of the exact value?

We know that the error of the degree-n Taylor approximation centred at $x=c$ of $f(x)$ at x_0 is:

$$E_n(x_0) = \frac{f^{(n+1)}(s)(x_0-c)^{n+1}}{(n+1)!}, \text{ for some } s \text{ between } x \text{ and } c$$

Taking $f(x) = \cos(x)$, $c = 0$, and $x_0 = 0.5$, we have:

$$|E_n(0.5)| = \left|\frac{f^{(n+1)}(s)(0.5-0)^{n+1}}{(n+1)!}\right| \leq \left|\frac{(0.5)^{n+1}}{(n+1)!}\right|$$

So, we want to know for what n is $\frac{1}{2^{n+1}(n+1)!} < \frac{1}{2^8(8!)}$.

This happens for $n \geq 8$. So the mimimal such n is 8.

QUIZ 7

First name: Observed Common Mistakes

Last name:

Student number:

Recitation section:

1. Calculate the Taylor polynomial $T_4(x)$ centred at $x = \frac{\pi}{2}$ for $f(x) = \cos(x)$.

Overall, this was done well.

Remember that a table as I did above is a nice way to organize your work.

Also, your taylor polynomial is A POLYNOMIAL. If you have non-polynomial functions of x hanging around, you've done something wrong. The coefficients are $f^{(n)}(a)$, which is a constant.

Finally, don't forget that you have the $\frac{1}{n!}$ in the coefficent as well.

2. What is the minimal degree Taylor polynomial about $x = 0$ that you need to calculate $\sin(0.5)$ to strictly within $(2^{-8})/8!$ of the exact value?

A few major mistakes:

① When a problem says "strictly," this means you don't want the inequality equal to.

② If you're asked to find the error of the approximation at a particular value of x, this x needs to be in the error, as well as a centre that is close to that x.

③ Remember, the error formula says that there exists s between a and x such that $E_n(x) = \frac{f^{(n+1)}(s)(x-a)^{n+1}}{(n+1)!}$. You don't know what s is, but chances are you can bound $f^{(n+1)}(s)$.

④ You're asked to solve for n. You need to determine how big an approximation is necessary to achieve the desired accuracy.