Power Series 1, February 28, 2018

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1. Prove that $\sum_{n\geq 0} \frac{x^n}{n!}$ satisfies the differential equation $f'(x) = f(x)$

   See your class notes.

2. Find the power series representation for the following functions:

   SOLUTION NOTE: In all of the examples below, we use the convergence of the geometric series $\sum_{n\geq 0} u^n = \frac{1}{1-u}$, for $|u| < 1$. This means that the series converges to $\frac{1}{1-u}$ only when $|u| < 1$. We can use substituion of u (which may change the interval of convergence), as well as differentiation and integration (which do *not* change the interval of convergence) to find power series representations of the functions. That is, *we find a series that converges to the function for certain values of x.*

   (a)
   - $\frac{1}{2-x}$

     We have: $\frac{1}{2-x} = \frac{1}{2}\frac{1}{1-(\frac{x}{2})} = \frac{1}{2}\sum_{n\geq 0}(\frac{x}{2})^n = \sum_{n\geq 0}\frac{x^n}{2^{n+1}}$, for $|\frac{x}{2}| < 1$
   - $\frac{1}{x^2+1}$

     We have: $\frac{1}{x^2+1} = \frac{1}{1-(-x^2)} = \sum_{n\geq 0}(-x^2)^n = \sum_{n\geq 0}(-1)^n x^{2n}$, for $|x^2| < 1$
   - $\frac{3}{1-2x^2}$

     We have: $\frac{3}{1-2x^2} = 3\frac{1}{1-(2x^2)} = 3\sum_{n\geq 0}(2x^2)^n = 3\sum_{n\geq 0}2^n x^{2n}$, for $|2x^2| < 1$

   (b)
   - $\frac{1}{(1+x)^3}$

     We have: $\frac{1}{(1+x)^3} = \frac{1}{2}\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{1}{1+x}\right)\right) = \frac{1}{2}\frac{d}{dx}\left(\frac{d}{dx}\left(\sum_{n\geq 0}(-x)^n\right)\right) = \frac{1}{2}\frac{d}{dx}\left(\frac{d}{dx}\left(\sum_{n\geq 1}(-1)^n x^n\right)\right) =$
     $\frac{1}{2}\frac{d}{dx}\left(\sum_{n\geq 0}(-1)^n n x^{n-1}\right) = \frac{1}{2}\sum_{n\geq 2}(-1)^n n(n-1)x^{n-2}$, which will be true for $|x| < 1$
   - $\frac{1}{(1-x)^3}$

     We have: $\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{1}{1-x}\right)\right) = \frac{1}{2}\frac{d}{dx}\left(\frac{d}{dx}\left(\sum_{n\geq 0}(x)^n\right)\right) = \frac{1}{2}\frac{d}{dx}\left(\sum_{n\geq 0}n x^{n-1}\right) =$
     $\frac{1}{2}\sum_{n\geq 2}n(n-1)x^{n-2}$, which will be true for $|x| < 1$

- $\arctan(x)$

  We have that $\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \sum_{n\geq 0}(-t^2)^n = \int_0^x \sum_{n\geq 0}(-1)^n t^{2n} = \sum_{n\geq 0}\frac{(-1)^n x^{2n+1}}{2n+1}$.

  This equality holds for $|x| < 1$

(c)

- $\frac{x}{x^2+1}$

  We have that $\frac{x}{x^2+1} = x\frac{1}{x^2+1} = \frac{1}{1-(-x^2)} = \sum_{n\geq 0}(-x^2)^n = \sum_{n\geq 0}(-1)^n x^{2n}$, for $|x^2| < 1$

- $\frac{1+x}{1-x}$

  We have that $\frac{1+x}{1-x} = \frac{1}{1-x} + \frac{x}{1-x} = \sum_{n\geq 0} x^n + x\sum_{n\geq 0} x^n = 1 + 2\sum_{n\geq 1} x^n$, for $|x| < 1$

- $\frac{2x+3}{x^2+6x+2}$

  This one is a little harder than the others because if we want to use the geometric series, we need to have it in the form of $\frac{1}{1-u}$ and we want to make sure that our result is a power series that is of the form $\sum_{n\geq 0} a_n(x-a)^n$. To do this, notice that $x^2+6x+2 = x^2+6x+9-9+2 = (x+3)^2 - 7 = -(7-(x+3)^2)$. Try and finish the reaminder of the problem yourself.