Taylor Series and Remainders, March 16, 2018

On Wednesday, we showed that the error of the Nth degree Taylor approximation of e^x, $\sin(x)$, $\cos(x)$, and $\log(1+x)$ tended to 0 as we took N to approach infinity. This means that the Taylor series of these functions actually converges to the function (where the series converges. The following Taylor series are ones you are expected to memorize for homeworks, quizzes, and exams. You are allowed to use them without proof.

1. $\dfrac{1}{1-x} = \sum_{n\geq 0} x^n$, for $|x| < 1$
2. $e^x = \sum_{n\geq 0} \dfrac{x^n}{n!}$, for all $x \in \mathbb{R}$
3. $\sin(x) = \sum_{n\geq 0} \dfrac{(-1)^n}{(2n+1)!} x^{2n+1}$, for all $x \in \mathbb{R}$
4. $\cos(x) = \sum_{n\geq 0} \dfrac{(-1)^n}{(2n)!} x^{2n}$, for all $x \in \mathbb{R}$
5. $\log(1+x) = \sum_{n\geq 1} \dfrac{(-1)^{n-1}}{n} x^n$, for $-1 < x \leq 1$
6. $\arctan(x) = \sum_{n\geq 0} \dfrac{(-1)^n}{2n+1} x^{2n+1}$, for $|x| < 1$

Remember that a few weeks ago, I said that it's significantly easier to determine whether a series converges than it is to determine what a series converges *to.* Here, we will play around with some series formulas and prove why they are true.

1. Prove that $\sum_{n\geq 1} \dfrac{1}{n!(n+2)} = \dfrac{1}{2}$

   We notice that the given series looks closest to $e^x = \sum_{n\geq 0} \dfrac{x^n}{n!}$, with $x = 1$. The only difference is that we have the $n+2$ in the denominator. What opperations would allow us to go from $\dfrac{x^n}{n!}$ to $\dfrac{x^{\square}}{n!(n+2)}$? I leave the exponent of the x as a square because we don't know what it will be just yet.

   We know that $\int_0^x t^{n+1} dt = \dfrac{x^{n+2}}{n+2}$. Unfortunately, for the power series of e^x, we only have x^n. How can we get x^{n+1}? We multiply by x. So, we will start with

   $$xe^x = \sum_{n\geq 0} \frac{x^{n+1}}{n!}$$

   Taking the integral from 0 to t of both sides yields:

   $$\int_0^t xe^x \, dx = \int_0^t \sum_{n\geq 0} \frac{x^{n+1}}{n!} \, dx$$
   $$te^t - e^t + 1 = \sum_{n\geq 0} \frac{t^{n+2}}{n!(n+2)}$$

Then, taking $t = 1$ and maling sure the sum starts at $n = 1$ gives us:

$$1e^1 - e^1 + 1 = \sum_{n\geq 0} \frac{1^{n+2}}{n!(n+2)}$$
$$1 = \frac{1}{2} + \sum_{n\geq 1} \frac{t^{n+2}}{n!(n+2)}$$

Hence,

$$\sum_{n\geq 1} \frac{1}{n!(n+2)} = \frac{1}{2}$$

2. Prove that $\sum_{n\geq 1} \frac{n}{(n+1)!} = 1$

   Similar to the last problem, let's take $f(x) = \frac{e^x - 1}{x}$. The Taylor series for this function is:

$$\frac{\sum_{n\geq 0} \frac{x^n}{n!} - 1}{x} = \frac{\sum_{n\geq 1} \frac{x^n}{n!}}{x} = \sum_{n\geq 1} \frac{x^{n-1}}{n!}$$

   .

   Now, taking derivatives of both sides yields (watch out for where the sum starts):

$$\frac{1}{x}e^x + (e^x - 1)(\frac{-1}{x^2}) = \sum_{n\geq 2} \frac{(n-1)x^{n-2}}{n!} = \sum_{n\geq 1} \frac{nx^{n-1}}{(n+1)!}$$

   Now, taking $x = 1$ yields:

$$\frac{1}{1}e^1 + (e^1 - 1)(\frac{-1}{1^2}) = \sum_{n\geq 1} \frac{n(1)^{n-1}}{(n+1)!} = \sum_{n\geq 1} \frac{n}{(n+1)!}$$

   Hence, $\sum_{n\geq 1} \frac{n}{(n+1)!} = 1$ as desired.

3. Prove that $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$. Weird, right?!

   HINT: Write $\sin(x)$ as a product of its roots and as a taylor series. Compare the coefficents. where do you see $\frac{1}{n^2}$?