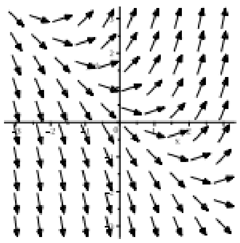
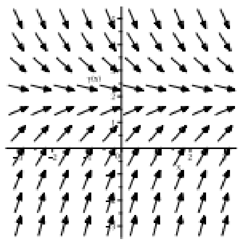
Differential Equations, March 21, 2018

Warm up: See class notes for solutions.

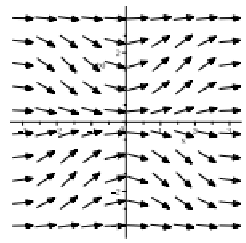
1. Match the follwoing direction fields with their corresponding differential equation.



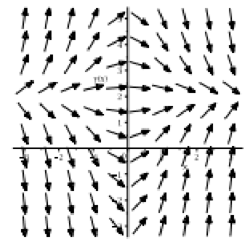
(I)



(II)



(III)



(IV)

(a) $y' = 2-y$ (b) $y' = x(2-y)$ (c) $y' = x+y-1$ (d) $y' = \sin(x)\sin(y)$

Separable Differential Equations

1. $\dfrac{dy}{dx} = \dfrac{\log(x)}{y}, y(1) = 3$

   solution: By separation of variables we have:

$$\begin{aligned} y\,dy &= \log(x)\,dx \\ \int y\,dy &= \int \log(x)\,dx \\ \frac{y^2}{2} + c &= x\log(x) - \int x\frac{1}{x}\,dx \\ \frac{y^2}{2} + c &= x\log(x) - x \end{aligned}$$

   Since $y(1) = 3$, we have that $\dfrac{3^2}{2} + c = 1\log(1) - 1$, so $c = -1 - \dfrac{9}{2}$. Hence, $\dfrac{y^2}{2} - \dfrac{11}{2} = x\log(x) - x$

2. $\dfrac{dy}{dx} = e^y \sin(x), y(0) = 1$

   solution: By separation of variables we have:

$$\begin{aligned} \frac{dy}{e^y} &= \sin(x)\,dx \\ \int \frac{dy}{e^y} &= \int \sin(x)\,dx \\ -e^{-y} + c &= -\cos(x) \end{aligned}$$

   Since $y(0) = 1$, we have that $-e^{-1} + c = -\cos(0)$, so $c = -1 + \dfrac{1}{e}$. Hence, $-e^{-y} + -1 + \dfrac{1}{e} = -\cos(x)$

3. $\dfrac{dy}{dx} = \sqrt{xy}, y(1) = 2$

$$\frac{dy}{\sqrt{y}} = \sqrt{x}\,dx$$
$$\int \frac{dy}{\sqrt{y}} = \int \sqrt{x}\,dx$$
$$2y^{1/2} + c = \frac{x^{3/2}}{3/2}$$

Since $y(1) = 2$, we have that $2(3^{1/2}) + c = \frac{0^{3/2}}{3/2}$. Hence, $y^{1/2} = \frac{x^{3/2}}{3} + \sqrt{3}$

The infamous draining tank problem:

Suppose a reservoir contains 1000 L of polluted water, in which is dissolved 10 kg of pollutant. Suppose polluted water flows into the tank at a rate of 10 L/min, with 0.1 kg of pollutant per litre. The reservoir is contatnly and thoroughly mixed, and the solution flows out at a rate of 10 L/min.

1. Find a function describing the amount of pollutant in the resevoir at time t.

   solution:Here, it's first important to realize that the rate of change of the pollutant in the tank is equivalent to the rate in minus the rate out. So, we let $P(t)$ be the amount of pollutant in the tank at time t. So, $\frac{dP}{dt}$ is the rate of change in the pollutant in the tank with respect to time. We have that $\frac{dP}{dt} = Rate\,in - rate\,out$. We have the rate of pollutant in is $\frac{10L}{min}\frac{0.1kg}{L}$. Similarly, the rate of pollutant out is $\frac{10L}{min}\frac{P}{100L}$. So:

$$\frac{dP}{dt} = 0.1 - \frac{P}{10}$$
$$\frac{dP}{0.1 - 0.1P} = dt$$
$$\int \frac{dP}{0.1 - 0.1P} = \int dt$$
$$-10\log|0.1 - 0.1P| + c = t$$
$$P = 1 + Ce^{-0.1t}$$

   Since the amount of pollutant initially is 10 kg at $t = 0$, we have that $c = 9$. So, $P = 1 + 9e^{-0.1t}$

2. Describe what happens as t gets very large. Does your mathematical model conform with your physical intuition?

   solution: As t gets large, we see that P approaches 1.

3. Repeat parts (1) and (2), assuming that the water flowing into the resevoir is pure.

   solution: The only difference here will be that $P_{in} = 0$.