Differential Equations III, March 28, 2018

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1. WARMUP: See class notes for solution
   - Determine the family of solutions to the differential equation $y' + 2y = e^{t/2}$,
   - Given the initial condition of $y(1) = 2$, what is the solution to the differential equation?
   - Draw a direction field and a few solutions' graphs.
   - Summarize the method of integrating factors
2. A tank has pure water flowing into it at 12 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Initially, the tank contains 10 kg of salt in 100 L of water. Find the amount of salt in the tank after t minutes.

   SOLUTION: Volume of the water in the tank is not constant. We can define $V = 100 + 2t$. So, the concentration of salt, S, after t minutes is $\frac{S}{V} = \frac{S}{100+2t}$. Hence, the rate of change of the amount of salt with respect to time is $\frac{dS}{dt} = S_{in} - S_{out} = 0 - \frac{S}{100+2t}(10L/min)$. Solving this separable differential equation yields $S = \frac{10^{11}}{(100+2t)^5}$
3. Find the orthogonal trajectories of the family of curves defined by $y^2 = kx^3$.
   SOLUTION: We need to determine the curves that are perpendicular to the family of curves at points (x, y). In order to do this, we need the slope of the family of curves. Taking d/dx of both sides yeilds $2y\frac{dy}{dx} = 3kx^2$, so $\frac{dy}{dx} = \frac{3kx^2}{2y}$. Since $k = \frac{y^2}{x^3}$, $\frac{dy}{dx} = \frac{3y}{2x}$. Since we want the slope of our orthogonal trajectories to be perpendicular to this, we want the slopes to be the inverse reciprocal, so $\frac{dy}{dx} = -\frac{2x}{3y}$. Now, this is a separable differential equation and we can solve for y. After a little bit of work, we get $y^2 = -\frac{2}{3}x^2 + \frac{2}{3}C$, for some constant C. Then, $\frac{y^2}{\frac{2}{3}C} + \frac{\frac{2}{3}x^2}{\frac{2}{3}C} = 1$, which is equivalent to $\frac{y^2}{\left(\sqrt{\frac{2C}{3}}\right)^2} + \frac{x^2}{(\sqrt{C})^2} = 1$,

   which is a family of ellipses. The trajectories look as follows: