Power Series 2, March 2, 2018

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1. Suppose that the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$ is R. What is the radius of convergence of the power series $\sum_{n\geq 0} a_n x^{2n}$?

   Solution: There are three cases that need to be consider. Recall that given a power series $\sum_{n\geq 0} a_n(x-a)^n$, the interval of convergence is either 1. All real numbers 2. Only at $x = a$ or 3. on $(a - R, a + R)$, where $R > 0$.

   So, we can deal with each of the cases individually.

   - If $f(x) = \sum_{n\geq 0} a_n x^n$ converges for all real numbers, then $f(x^2) = \sum_{n\geq 0} a_n(x^2)^n$ converges for all values of x.
   - If $f(x) = \sum_{n\geq 0} a_n x^n$ converges only at $x = 0$, then $f(x^2) = \sum_{n\geq 0} a_n(x^2)^n$ must only converge at $x = 0$.
   - If $f(x) = \sum_{n\geq 0} a_n x^n$ converges for $(-R, R)$, this means it converges for $|x| < R$. So, $f(x^2) = \sum_{n\geq 0} a_n(x^2)^n$ converges for $|x^2| < R$, which means it converges for $|x| < \sqrt{R}$.

2. Find the power series representation and interval of convergence for $\frac{x^3}{(2-x)^2}$.

   Solution:
   $$\frac{x^3}{(2-x)^2} = x^3\frac{d}{dx}(\frac{1}{2-x}) = \frac{x^3}{2}\frac{d}{dx}\left(\sum_{n\geq 0}(\frac{x}{2})^n\right) = \frac{x^3}{2}\left(\sum_{n\geq 1}\frac{nx^{n-1}}{2^n}\right) = \sum_{n\geq 1}\frac{n}{2^{n+1}}x^{n+2}, \text{ for } |x| < 2$$

3. Find the sums of the following series:

   (a) $\sum_{n\geq 1} nx^{n-1}$

   We note that the terms inside the sum $nx^{n-1} = \frac{d}{dx}x^n$.

   So, $\sum_{n\geq 1} nx^{n-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

   (b) $\sum_{n\geq 1} nx^n$

   We note that the terms inside the sum $nx^n = x\frac{d}{dx}x^n$.

   So, $\sum_{n\geq 1} nx^n = x\sum_{n\geq 1} nx^{n-1}x\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$

   (c) $\sum_{n\geq 1}\frac{n}{2^n}$

This series looks just like the power series in part b, with $x = \frac{1}{2}$. So, since $\sum_{n\geq 1} nx^n = \frac{x}{(1-x)^2}$, we have that $\sum_{n\geq 1} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2$

(d) $\sum_{n\geq 2} \frac{n(n-1)}{2^n}$

HINT: $n(n-1)x^n = n(n-1)x^{n-1}x^2$

(e) $\sum_{n\geq 1} \frac{n^2}{2^n}$

HINT: $\frac{d}{dx} nx^n = n^2 x^{n-1}$.