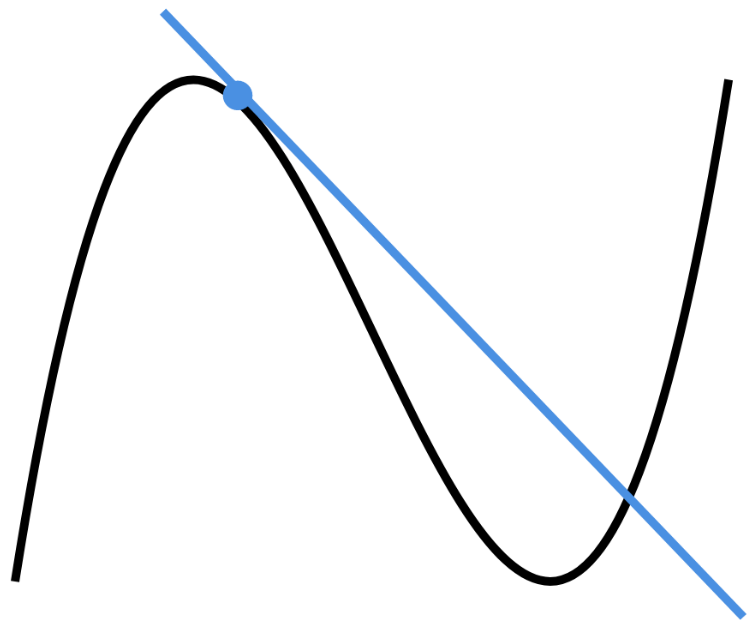


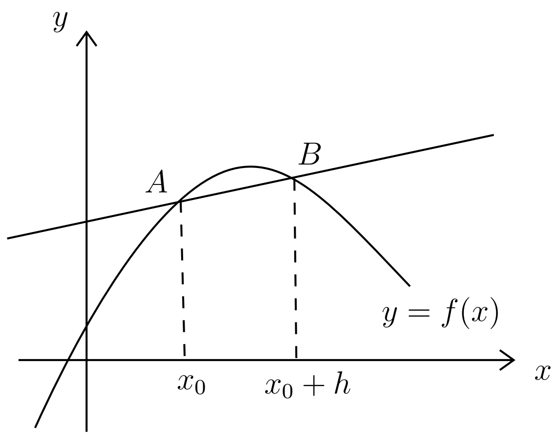
MATH 100 Differential Calculus with One Variable

Wucheng Zhang

http://blogs.ubc.ca/wucheng/

Introduction

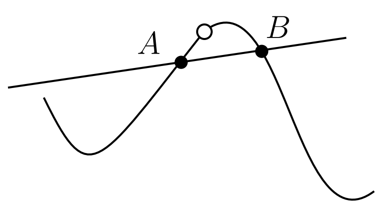
The big question in this course to find the tangent line of a curve $y = f(x)$ at $x = x_0$. In Latin, "tangent" means "to just touch", i.e. to have only one cross point. Therefore, we apply a dynamic approach. Pick up two points $A : (x_0, f(x_0))$ and $B : (x_0 + h, f(x_0 + h))$ which the line is across.



To get one point, we want to push the point B approaching to A, i.e. take the limit $h \to 0$. Then we could find out the slope of the tangent line

$$k_{\text{slope}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

This requires us to further explore what happens with this approaching process - which is **LIMIT**. Besides, when B is approaching to A, we may the occur jumps and holes.



Bad? Good?

To deal with this kind of situation, we need the concept of **CONTINUITY** and the limit happens to be the powerful tool examine to this situation. Then we are able to define and calculate the slope of the tangent line, which we define as **DERIVATIVE**. Based on the definition, we could explore a set of theorems including **MEAN VALUE THEOREM**. With all the knowledge set up, we could **APPLY** them in curve sketching and optimization.

Contents

Limit

1. **Definition**

Let's consider the two examples below:

First consider the sequence $a_n = \dfrac{n}{n+1}$. With few terms listed out

$$\{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}$$

we could notice that a_n is approaching to 1 - the distance between 1 and a_n, $|a_n - 1| = \dfrac{1}{n+1}$, is getting smaller. We could make this feeling more mathematical . Let the error $\varepsilon_1 = \dfrac{1}{10}$, we want to have

$$|a_n - 1| = \frac{1}{n+1} < \frac{1}{10}$$

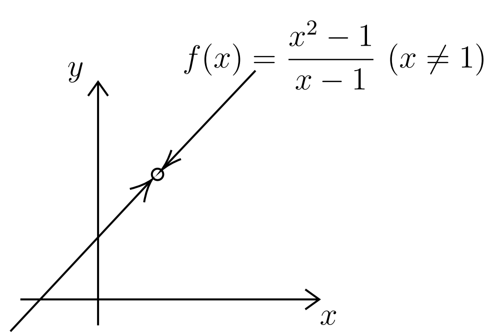
which implies $n > 9$. Let the error $\varepsilon_2 = \dfrac{1}{100}$, we want to have

$$|a_n - 1| = \frac{1}{n+1} < \frac{1}{100}$$

which implies $n > 99$. In general, for any error $\varepsilon > 0$, we want to have $|a_n - 1| = \dfrac{1}{n+1} < \varepsilon$, then we need $n > [\dfrac{1}{\varepsilon}] - 1$. This also gives a precise description that the a_n is approaching to 1 while n is getting bigger. We want to abstract this kind of approaching as limit, $\lim\limits_{n\to\infty} a_n = 1$. Therefore, we could give the definition of sequence limit.

> **Definition 1.** If for any given $\varepsilon > 0$, there exists $N > 0$, when $n > N$, we have $0 < |a_n - A| < \varepsilon$ where A is a constant and a_n is a sequence, then we know the limit of a_n is A and we could denote it as $\lim\limits_{n\to\infty} a_n = A$.

Then we consider the function $f(x) = \dfrac{x^2-1}{x-1}\,(x \neq 1)$ which is equivalent to $f(x) = x + 1\,(x \neq 1)$



We have the following observation. While x is approaching to 1 from left hand side, $x \to 1^-$, $f(x)$ is approaching 2. While x is approaching to 1 from right hand side, $x \to 1^+$, $f(x)$ is approaching 2. Since "approaching" does not require reaching, we could interpret "approaching" as $x \neq a$ but $x \to a^+$ and $x \to a^-$. In this way, we could say while x is approaching to 1, $f(x)$ is approaching to 2. Inspired by Definition 1, we could give a more precise description. For any given tolerated error ε, we want $\left|\dfrac{x^2-1}{x-1}-2\right| = |x-1| < \epsilon$. Then let $\delta < \varepsilon$, when $0 < |x-1| < \delta$, $\left|\dfrac{x^2-1}{x-1}-2\right| < \varepsilon$. Then we could give the definition of function limit.

Definition 2. If for any given $\varepsilon > 0$, there exists $\delta > 0$, when $0 < |x-a| < \delta$, we have $0 < |f(x) - A| < \varepsilon$ where A is a constant and $f(x)$ is a function with domain D, then we know the limit of $f(x)$ is A and we could denote it as $\lim\limits_{x\to a} f(x) = A$.

Summarize the two examples above, we could find that the approaching of x could go to a number or to infinity (n could only go to infinity because of the discreteness) and $f(x)$ or a_n could go to a number or to infinity as well (However, goes to infinity means the limit does not exist). Then we could give two more generalized definitions of sequence and function limit.

Definition 3. If for any $\underline{(1)}$, there exists $\underline{(2)}$, when $\underline{(3)}$, we have $\underline{(4)}$, then $\lim\limits_{\underline{(5)}} \underline{(7)} = \underline{(6)}$.

For sequences, we have (7) to be a_n

(2)	(3)	(5)
$N > 0$	$n > N$	$n \to \infty$

and

(1)	(4)	(6)
$\varepsilon > 0$	$0 < \lvert a_n - A\rvert < \varepsilon$	A
$M > 0$	$a_n > M$	$+\infty$
$M > 0$	$a_n < -M$	$-\infty$
$M > 0$	$\lvert a_n\rvert > M$	$\pm\infty$

For functions, we have (7) to be $f(x)$

(2)	(3)	(5)
$\delta > 0$	$0 < \lvert x-a\rvert < \delta$	$x \to a$
$\delta > 0$	$0 < x-a < \delta$	$x \to a^+$
$\delta > 0$	$-\delta < x-a < 0$	$x \to a^-$
$X > 0$	$x > X$	$x \to \infty$
$X > 0$	$x < -X$	$x \to -\infty$
$X > 0$	$x > \lvert X\rvert$	$x \to \pm\infty$

and

(1)	(4)	(6)
$\varepsilon > 0$	$0 < \lvert f(x) - A\rvert < \varepsilon$	A
$M > 0$	$f(x) > M$	$+\infty$
$M > 0$	$f(x) < -M$	$-\infty$
$M > 0$	$\lvert f(x)\rvert > M$	$\pm\infty$

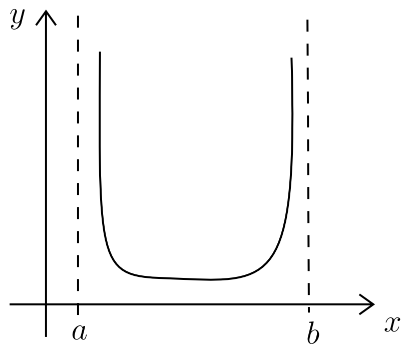
Let's check an example.

Example 1. Prove $\lim_{x \to 4} \sqrt{x} = 2$.

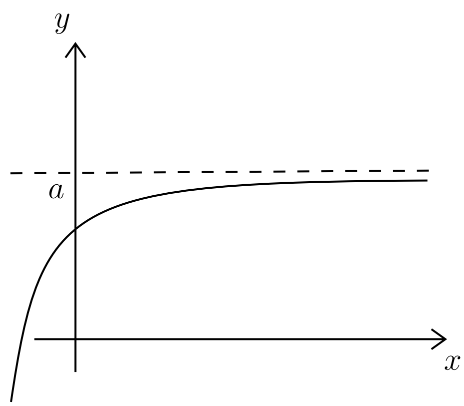
Proof. Since $|\sqrt{x} - \sqrt{4}| = \left| \dfrac{x-4}{\sqrt{x}+2} \right| < \dfrac{1}{2}|x-4| = \dfrac{\delta}{2} = \varepsilon$, we let $\delta = 2\varepsilon$. Then for any given $\varepsilon > 0$, we can find $\delta = 2\varepsilon$ such that when $0 < |x-4| < \delta$, $|\sqrt{x} - 2| < \varepsilon$. Therefore, $\lim_{x \to 4} \sqrt{x} = 2$.

Besides the existence of limit, we have to notice that we can only call A to be the limit of a_n or $f(x)$ when (6) is A where constant A exists. So if we want to show, i.e. $\underline{\lim_{x \to a}} \, \underline{f(x)} \neq A$, we have to show $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists x$ such that $0 < |x-a| < \delta$, $|f(x) - A| > \varepsilon$. If we want to show, i.e. $\underline{\lim_{x \to a}} \, \underline{f(x)}$ does not exists, we have to show $\forall A \in \mathbb{R}$, $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists x$ such that $0 < |x-a| < \delta$, $|f(x) - A| > \varepsilon$.

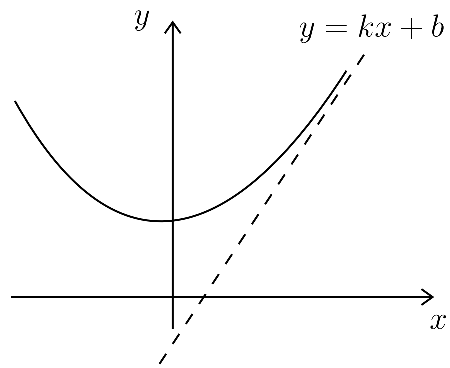
Based on that, we can also make some discussion on asymptotes. If we have $\lim_{x \to a} f(x) = \pm\infty$ and , we have $x = a$ and $x = b$ to be the *vertical asymptotes*.



It is obvious that there could be multiple vertical asymptotes for a function $f(x)$. If we have $\lim_{x \to \pm\infty} f(x) = a$, we have $y = a$ to be the *horizontal asymptote*.



More generally, if we have $\lim_{x \to \infty} [f(x) - (kx + b)] = 0$, then we have $y = kx + b$ to the *oblique asymptote*.



2. **Properties**
   (*Note:* In the following content, a domain D should either be $|x-a| < \delta$ or $x > X$.)
   (a) **Calculation Properties** Assuming $\lim f = A$ and $\lim g = B$ exist (the approaching of x or n are the same):
      i. $\lim[f \pm g] = A \pm B$
      Proof. By definition, we know $\forall \varepsilon_1 > 0$, $\exists D_1$ such that when $x \in D_1$, $0 < |f - A| < \varepsilon_1$ and $\forall \varepsilon_2 > 0$, $\exists D_2$ such that when $x \in D_2$, $0 < |g - B| < \varepsilon_2$. Let $D = D_1 \cap D_2$ (by the definition of limit we know $D \neq \varnothing$) and $\varepsilon_1 = \varepsilon_2 = \dfrac{\varepsilon}{2}$. Then when $x \in D$, we have
$$0 < |(f+g) - (A+B)| \leq |f - A| + |g - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
      by trigonometry inequality. Therefore $\lim[f+g] = A + B$. And then when $x \in D$, we have
$$0 < |(f-g) - (A-B)| \leq |f - A| + |g - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
      by trigonometry inequality. Therefore $\lim[f-g] = A - B$.

      ii. $\lim fg = AB$
      Proof. Let $\varepsilon > 0$. By definition of the limits of g and f, $\exists D_1$ such that when $x \in D_1$, $|f - A| < \sqrt{\dfrac{\varepsilon}{3}}$ (1). $\exists D_2$ such that when $x \in D_2$, $|f - A| < \dfrac{\varepsilon}{3B}$ (2). $\exists D_3$ such that when $x \in D_3$, $|g - B| < \sqrt{\dfrac{\varepsilon}{3}}$ (3). $\exists D_4$ such that when $x \in D_4$, $|g - B| < \dfrac{\varepsilon}{3A}$ (4). Take $D = D_1 \cap D_2 \cap D_3 \cap D_4$, when $x \in D$, we would have (1)(2)(3)(4) all are satisfied. Since
$$(f - A)(g - B) = fg - Ag - Bf + AB$$

which comes out to be

$$\begin{aligned}|fg - AB| &= |(f-A)(g-B) + Ag + Bf - 2AB| \\ &= |(f-A)(g-B) + B(f-A) + A(g-B)| \\ &< |f-A||g-B| + B|f-A| + A|g-B| \\ &= \sqrt{\frac{\varepsilon}{3}}\sqrt{\frac{\varepsilon}{3}} + B\frac{\varepsilon}{3B} + A\frac{\varepsilon}{3A} \\ &= \varepsilon\end{aligned}$$

So $\forall \varepsilon > 0$, we find the D such that when $x \in D$, $|fg - AB| < \varepsilon$. By definition $\lim fg = AB$.

iii. $\lim \frac{f}{g} = \frac{A}{B}\,(B \neq 0)$

Proof. First we are going to prove a lemma $\lim \frac{1}{g} = \frac{1}{B}$. Let $\varepsilon > 0$. By definition, we know that there is a D_1 such that when $x \in D_1$, $|g - B| < \frac{|B|}{2}$. Therefore, $|\,|B| - |g|\,| \leq |g - B| < \frac{|B|}{2}$ by triangle inequality which implies $|g| > \frac{|B|}{2}$ and then implies $0 < \frac{1}{|g|} < \frac{2}{|B|}$. Also, by definition there is D_2 such that when $x \in D_2$, we have $0 < |g - B| < \frac{|B|^2}{2}\varepsilon$. Let $D = D_1 \cap D_2$. Then when $x \in D$, we have

$$\begin{aligned}|\frac{1}{g} - \frac{1}{B}| = \frac{|g-B|}{|gB|} &= \frac{1}{|g|}\frac{1}{|B|}|g-B| \\ &< \frac{2}{|B|}\frac{1}{|B|}\frac{|B|^2}{2}\varepsilon = \varepsilon\end{aligned}$$

and also $|\frac{1}{g} - \frac{1}{B}| = \frac{1}{|g|}\frac{1}{|B|}|g - B| > 0$. Therefore, we have $\lim \frac{1}{g} = \frac{1}{B}$. By the calculate property ii, we have $\lim \frac{f}{g} = \frac{A}{B}$.

iv. If $\lim\limits_{u \to a} f(u) = A$ and $\lim\limits_{x \to x_0} \varphi(x) = a$, then we have $\lim\limits_{x \to x_0} f[\varphi(x)] = A$

Proof. Let $\varepsilon > 0$, by definition, there is a $\eta > 0$ such that when $0 < |u - a| < \eta$, $0 < |f(u) - A| < \varepsilon$. For $\eta > 0$, by definition, we have there exists a $\delta > 0$ such that when $0 < |x - x_0| < \delta$, $0 < |\varphi(x) - a| < \eta$. Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $0 < |x - x_0| < \delta$, $0 < |f(g(x)) - A| < \varepsilon$. Therefore, $\lim\limits_{x \to x_0} f[g(x)] = A$.

(b) **General Properties**

i. **Uniqueness**

The limit is unique, i.e. if $\lim f = A$ and $\lim g = B$, then $A = B$.

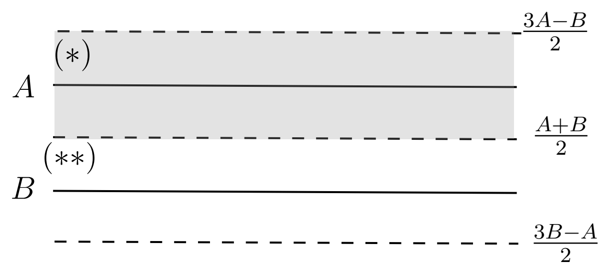
Proof. We are going to prove it by contradiction. Assume $A > B$ then let $\varepsilon = \frac{A-B}{2} > 0$. By definition, there exists D_1 such that when $x \in D_1$ we have $|f(x) - A| < \frac{A-B}{2}$ which is

$$\frac{A+B}{2} < f(x) < \frac{3A-B}{2} \tag{*}$$

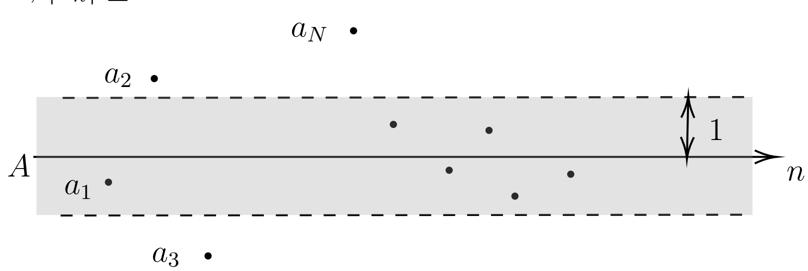
And also by definition, there exists D_2 such that when $x \in D_2$ we have $|f(x)-B| < \frac{A-B}{2}$ which is

$$\frac{3B-A}{2} < f(x) < \frac{A+B}{2} \tag{**}$$

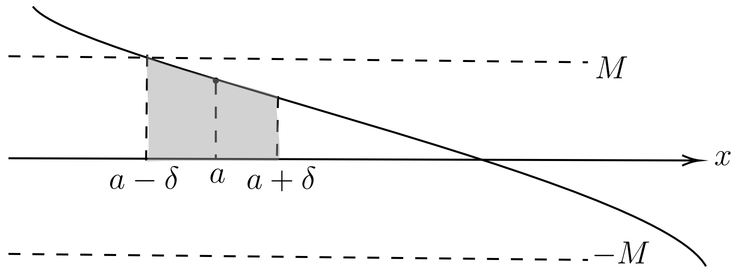
Let $D = D_1 \cap D_2$. When $x \in D$, (*) and (**) conflict. By contradiction $A \leq B$. By symmetric $A \geq B$. Therefore, $A = B$.
We could show this in the graph.



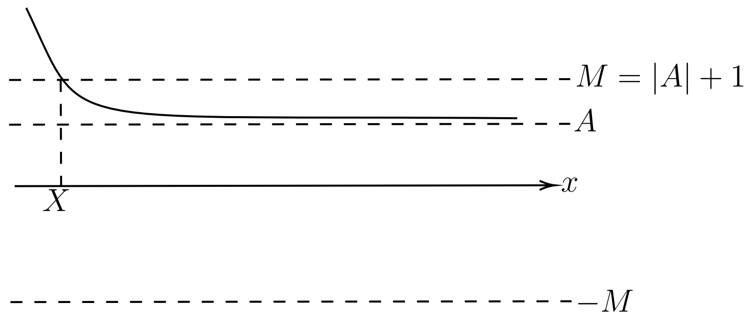
ii. **Boundedness**
(Sequence) If $\lim_{n\to\infty} a_n = A$, there exists $M > 0$ such that $|a_n| \leq M$.
Proof. Let $\varepsilon = 1$. By definition, we can find a $N > 0$ such that when $n > N$, $|a_n - A| < 1$. Therefore, we know, by triangle inequality, $||a_n| - A| \leq |a_n - A| < 1$ which implies $|a_n| < 1 + A$. Let $M = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |A|\}$. Then we have for all n, $|a_n| \leq M$.



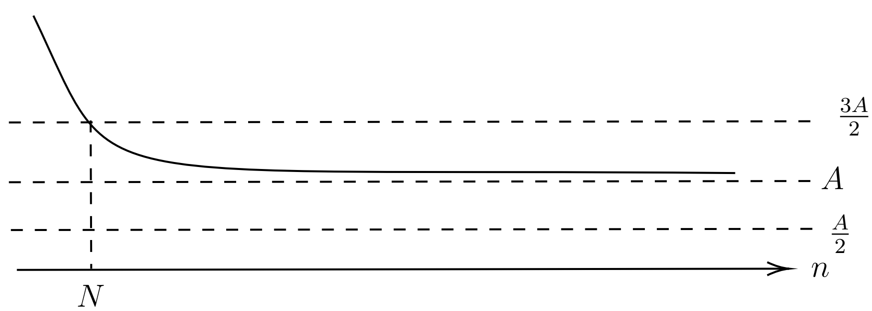
(Function 1 - Partially Bounded) If $\lim_{x\to a} f(x) = A$, then there exists $\delta > 0$ and $M > 0$ such that when $0 < |x - a| < \delta$, $|f(x)| \leq M$.
Proof. Let $\varepsilon = 1$. By definition, we can find a $\delta > 0$ such that when $0 < |x-a| < \delta$, $|f(x) - A| < 1$ which implies $|f(x)| < 1+|A|$. Let $M = 1+|A| > 0$. Then we have when $0 < |x - a| < \delta$, $|f(x)| \leq M$



(Function 2 - Partially Bounded) If $\lim\limits_{x\to\infty} = A$, there exists $X > 0$ and $M > 0$ such that when $x > X$, $|f(x)| \leq M$.
Proof. Let $\varepsilon = 1$. By definition, we can find an $X > 0$, when $x > X$, $|f(x) - A| < 1$ which implies $|f(x)| < 1 + |A|$. Let $M = 1 + |A| > 0$. When $x > X$, $|f(x)| \leq M$.

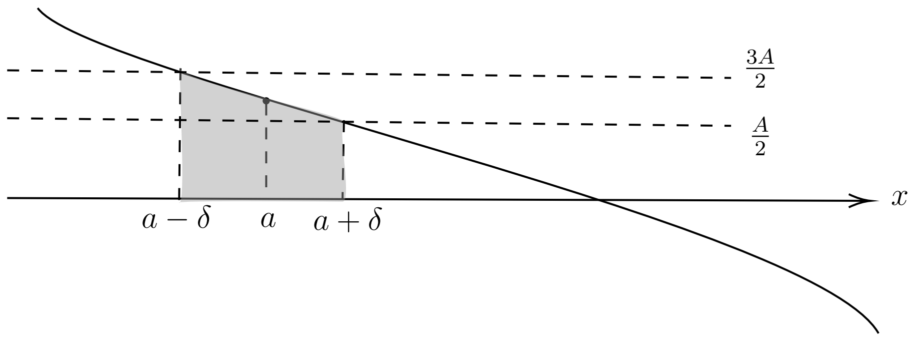


iii. **Sign-Preserving**
(Sequence) If $\lim\limits_{n\to\infty} a_n = A > 0 (< 0)$, there exists $N > 0$ such that when $n > N$, $a_n > 0 (< 0)$.
Proof. Let $A > 0$. Let $\varepsilon = \dfrac{A}{2} > 0$. By definition, we can find a $N > 0$ such that when $n > N$, $|a_n - A| < \dfrac{A}{2}$ which implies $a_n > \dfrac{A}{2} > 0$. We can prove the $A < 0$ case by the same method.



(Function 1) If $\lim\limits_{x\to a} f(x) = A > 0 (< 0)$, there exists a $\delta > 0$, when $0 < |x - a| < \delta$, we have $f(x) > 0 (< 0)$.
Proof. Let $A > 0$. Let $\varepsilon = \dfrac{A}{2} > 0$. By definition, we can find a $\delta > 0$ such that

when $0 < |x-a| < \delta$, $|f(x)-A| < \dfrac{A}{2}$ which implies $f(x) > \dfrac{A}{2} > 0$. We can prove the $A < 0$ case by the same method.



Corollary: If $\lim\limits_{x\to a} f(x) = L > A(< A)$, there exists a $\delta > 0$, when $0 < |x-a| < \delta$, we have $f(x) > A(< A)$.

Proof. Construct $g(x) = f(x) - A$. Then from $\lim\limits_{x\to a} f(x) - A = L - A > 0(< 0)$, we have $f(x) - A > 0(< 0)$ which is $f(x) > A(< A)$.

(c) **Existence Properties**

i. **Squeeze Theorem**

If $f(x) < g(x) < h(x)$ and $\lim f(x) = \lim h(x) = A$, then $\lim g(x) = A$.

Proof. For $\forall \varepsilon > 0$, $\exists D_1$ such that $x \in D_1$,

$$A - \varepsilon < f(x) < A + \varepsilon \tag{*}$$

and $\exists D_2$ such that $x \in D_2$,

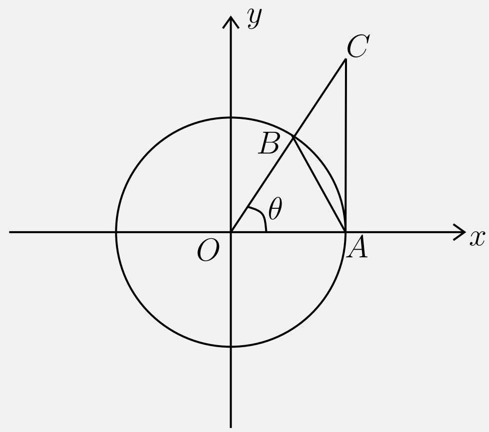
$$A - \varepsilon < h(x) < A + \varepsilon \tag{**}$$

Take $D = D_1 \cap D_2$, when $x \in D$, (*) and (**) are true and

$$A - \varepsilon < f(x) < g(x) < h(x) < A + \varepsilon$$

Then we know $\lim g(x) = A$

Example 2. Consider an important limit, $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.
Consider the following unit circle ($OA = 1$)



Denote the area of $\triangle AOB$ to be S_1, the area of sector AOB to be S_2 and the area of $\triangle AOC$ to be S_3. We know $S_1 = \frac{1}{2} \times \sin\theta \times 1 = \frac{1}{2}\sin\theta$, $S_2 = \frac{1}{2} \times \theta \times 1^2 = \frac{1}{2}\theta$ and $S_3 = \frac{1}{2} \times 1 \times \tan\theta = \frac{1}{2}\tan\theta$. Since $S_1 < S_2 < S_3$, we have

$$\sin\theta < \theta < \tan\theta$$

With arrangement, we have

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Since $\lim\limits_{\theta\to 0} 1 = 1$ and $\lim\limits_{\theta\to 0} \frac{1}{\cos\theta} = 1$, we have

$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$$

This is equivalent to, by changing dummy variables and flipping,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

ii. **Bounded Monotone Convergence Theorem**

Theorem 1. (Bounded Monotone Convergence Theorem) If $\{a_n\}$ is bounded and monotone, then a_n converges.

Proof. Without loss of generality, suppose $\{a_n\}$ is increasing and bounded. Let the least upper bound of $\{a_n\}$ is L. We want to prove $\{a_n\}$ converges to L. Given $\varepsilon > 0$, since $\{a_n\}$ is increasing, if none of $\{a_n\}$ eventually goes to $L - \varepsilon < a_n < L$, $L - \varepsilon$ would be the least upper bound, which be a contradiction. Therefore, all of $\{a_n\}$ eventually go to $L - \varepsilon < a_n < L$. Then we proved $\{a_n\}$ converges to L.

(d) **Other Properties**

Consider $f(x), g(x)$ where $|g(x)| \leq M$:

i. If $f(x)$ diverges to $\pm\infty$ and $g(x)$ does not converge to 0, then $f(x)g(x)$ diverges to $\pm\infty$.

ii. If $f(x)$ converges to 0, then $f(x)g(x)$ converges to 0.

iii. If $f(x)$ diverges to ∞ and $g(x)$ converges to 0, then $f(x)g(x)$ depends.

Problem Solving I

Prove the limit by definition; Calculate the limit or consider the convergence of the limit.

Problem 1: Prove the limit by definition

1. **Polynomial Function**. Prove the following limit converges or diverges (or does not not exist).
   (a) $\lim\limits_{n\to\infty} \dfrac{n-1}{n+1} = 1$
   (b) $\lim\limits_{x\to a} \sqrt{x} = \sqrt{a}$
   (c) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$
   (d) $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$
   (e) $\lim\limits_{n\to\infty} \dfrac{2n^2-1}{2n^2+1} = 1$
   (f) $\lim\limits_{x\to 2^+} \dfrac{1}{x^2-4} = +\infty$
   (g) Let $f(x) = \begin{cases} -\dfrac{3}{x+1},\ x < -1 \\ 4x+4,\ x > -1 \end{cases}$, then $\lim\limits_{x\to -1} f(x)$ does not exist.

2. **Discrete Function (Dirichlet Function)**. Prove the following limit exists or does not exist.
   (a) Let $f(x) = \begin{cases} 1,\ x = \dfrac{1}{10^k} \\ 0,\ x \neq \dfrac{1}{10^k} \end{cases}$, consider $\lim\limits_{x\to 0} f(x)$.
   (b) Let $f(x) = \begin{cases} 1,\ x \in \mathbb{Q} \\ 0,\ x \in \mathbb{R}/\mathbb{Q} \end{cases}$, consider $\lim\limits_{x\to a} f(x)$.
   (c) Let $f(x) = \begin{cases} x^2,\ x \in \mathbb{Q} \\ 0,\ x \in \mathbb{R}/\mathbb{Q} \end{cases}$, consider $\lim\limits_{x\to 0} f(x)$.
   (d) $\{a_n\} = \{0, 3, 0, 0, 3, 0, 0, 0, 3, \dots\}$, consider $\lim\limits_{n\to\infty} a_n$

3. **Trigonometric Function (Bounded Function)**. Prove the following limit converges or diverges (or does not not exist).
   (a) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$

Problem 2: Calculate the limit or consider the convergence of the limit

1. **Polynomial Function**. Calculate the following limit.

   (a) $P(x) = \dfrac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} = \begin{cases} \dfrac{a_n}{b_m} & n = m \\ 0 & n < m \\ \infty & n > m \end{cases}$

2. **Trigonometric Function (Bounded and Periodic Function)**. Calculate the following limits.

   (a) $\lim\limits_{n\to\infty} \cos(4n\pi) = 1$

   (b) $\lim\limits_{n\to\infty} \cos\left(\dfrac{3}{n}\right) = 0$

   (c) $\lim\limits_{n\to\infty} \dfrac{\sin 4n}{1+\sqrt{n}} = 0$

   (d) $\lim\limits_{x\to 0} x^2 \sin\left(\dfrac{1}{x}\right) = 0$

3. $\log_a n$, a^n, n^a, n^n, $n!$ **Functions**. ($n^n > n! > a^n > n^a > \log_a n$). Calculate the following limits.

   (a) $\lim\limits_{n\to\infty} \dfrac{(-5)^n}{n!} = 0$

   (b) $\lim\limits_{n\to\infty} \dfrac{n^n}{n!} = \infty$

   (c) $\lim\limits_{n\to\infty} (-1)^n \dfrac{2n}{\ln n}$ does not exist

4. **Special Type Functions:** $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\infty - \infty$. Calculate the following limit.

   (a) $\lim\limits_{t\to 1} \dfrac{t^3-1}{t^2-1} = \dfrac{3}{2}$

   (b) $\lim\limits_{t\to 1} \dfrac{t^3-t}{t^2-1} = 1$

   (c) $\lim\limits_{y\to 9} \dfrac{9-y}{3-\sqrt{y}} = 6$

   (d) $\lim\limits_{n\to\infty} (\sqrt{n^2+4n} - n) = 2$

   (e) $\lim\limits_{x\to\infty} (\sqrt{x^2+2} - x) = 0$

5. **Piecewise Function:**

   (a) $\lim\limits_{x\to 0} \dfrac{5x+12|x|}{5x-13x}$ does not exist

   (b) $\lim\limits_{x\to k} (x - [x])$ does not exist

6. **Application of Bounded-Monotone Convergence Theorem**. Calculate the limit or consider the convergence of the limit.

   (a) **Induction Sequence**

   - $a_{n+1} = \dfrac{1}{2}(a_n + 6)$ and $a_1 = 2$, prove $\lim\limits_{n\to\infty} a_n = 6$.
   - $a_{n+1} = \sqrt{2a_n}$ and $a_1 = \sqrt{2}$, prove $\lim\limits_{n\to\infty} a_n = 2$.
   - $a_{n+1} = \sqrt{a_n + 2}$ and $a_1 = \sqrt{2}$, prove $\lim\limits_{n\to\infty} a_n = 2$.

   (b) $\infty - \infty$ **Type:** Prove $\lim\limits_{n\to\infty} \sqrt{n^2 + 4n} - n$ converges.

   (c) **Constructed Sequences:**
   Let $\{a_n\}$ be a bounded sequence.Define a *crest* of the sequence to be an term am that is greater than all subsequent terms - that is, $a_m > a_n$ for all $n > m$.

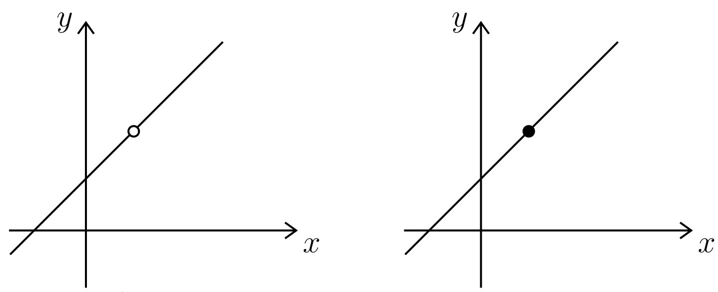
      i. Suppose $\{a_n\}$ has infinitely many crests. Prove that the crests form a convergent subsequence.

      ii. Suppose $\{a_n\}$ has only finitely many crests. Let a_{n_1} be a term with no subsequent crests. Construct a convergent subsequence with a_{n_1} as the first term.

Continuity

1. **Definition**

   Recall the example in the previous section $f(x) = \dfrac{x^2-1}{x-1}$ $(x \neq 1)$, we know $\lim\limits_{x\to 1^+} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1^-} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = 2$. These limits have nothing to do with the value of the function at $x = 1$. In other words, if we just know the limit $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = 2$, $f(1)$ could be any value. However, if we choose $f(1) = 2$. The right limit is approaching and touching the point $x = 1$ and the left limit is also approaching and touching the point $x = 1$. This means that they are connected.

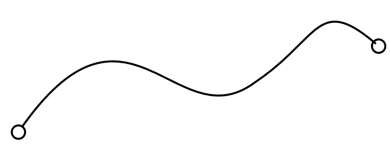
   

   $y = \frac{x^2-1}{x-1}$ $(x \neq 1)$ $\qquad$ $y = x + 1$

   In this example we could find $\lim\limits_{x\to a} f(x) = f(a)$ is a special case and instead of "connected", we want to use "continuous". Here, we give the definition of continuity.

   > **Definition 4.** A function $f(x)$ is *continuous* if $\lim\limits_{x\to a} f(x) = f(a)$. This means:
   >
   > - $f(x)$ is defined at $x = a$
   > - $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ exist and equal
   > - the three are equal

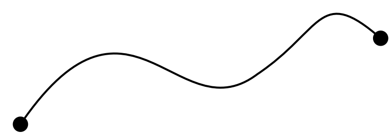
   Intuitively, we want to extend the definition from one point to a interval. It would be clear on an open interval.

   Since it is an open for every point in the interval we could always find the points in left and the points in the right. We could easily requires that every point in the interval is continuous.

Definition 5. A function $f(x)$ is continuous on (ℓ, r) if for any $a \in (\ell, r)$, $\lim_{x \to a^+} f(x) = f(a)$.

However, things become tricky for a closed interval $[\ell, r]$. We could see there is no point to the left of $x = \ell$ and there is not point to the right of $x = r$. Therefore, we could not find the left limit of $x = \ell$ and the right limit of $x = r$.



Then we just let the right limit of $x = \ell$ equals $f(\ell)$ and the left limit of $x = r$ equals $f(r)$.

Definition 6. A function $f(x)$ is continuous on $[\ell, r]$ if

- for any $a \in (\ell, r)$, $\lim_{x \to a^+} f(x) = f(a)$,
- $\lim_{x \to \ell^+} f(x) = f(\ell)$,
- $\lim_{x \to r^-} f(x) = f(r)$.

Based on the definition, we could explore the fact that the following functions are all continuous on their domain: polynomials, rational functions, rational powers, sums, products, quotients and composites of continuous functions, trigonometric functions and exponential functions.

2. **Theorems**

There are three theorems consequent to the the continuity.

(a) **Boundness Theorem**

Theorem 2. (Boundness Theorem) If $f(x)$ is continuous on $[\ell, r]$, $f(x)$ is bounded on $[\ell, r]$.

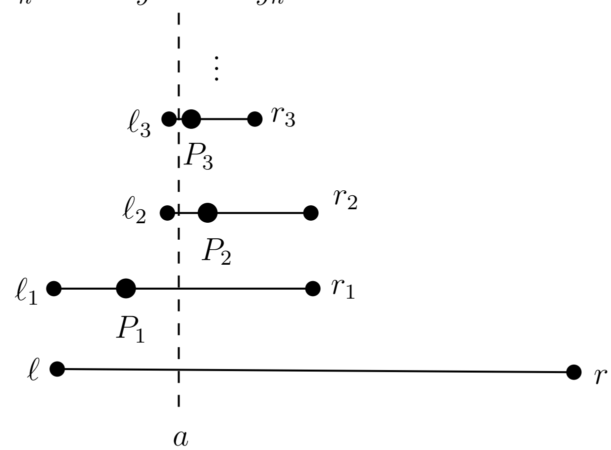
Proof. (Adapted from the notes by Dr. Fok-shuen Leung) We are going to prove it by contradiction. Without loss of generality, we assume that $f(x)$ has no upper bound.

Then we cut the interval $[\ell, r]$ into half and pick up the half with no upper bound (there should be at least one half with no upper bound and if there is two, pick the left half). We denote the picked half as $[\ell_1, r_1]$ and we pick a point in $[\ell_1, r_1]$ whose y-value is greater then 1 to be P_1. Then with the same method we could find the second the interval $[\ell_1, r_1]$ and the point $P_2 \in [\ell_1, r_1]$ whose y-value is greater then 2.

In this procedure, we obtain a list of interval

$$[\ell, r] \supset [\ell_1, r_1] \supset [\ell_2, r_2] \supset \cdots$$

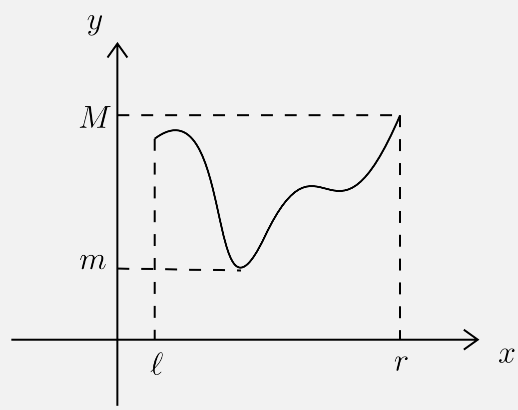
and a list of points P_n whose y-values $y_n > n$.



The interval $[\ell_n, r_n]$ would converges to one point a. Since it is continuous, $\lim_{x\to a} f(x) = f(a)$. However, if we take a $\varepsilon > 0$, no matter how small the interval $(a-\delta, a+\delta)$ is, we can always find some points P_k, $P_{k+1}, \ldots$ where $k > f(a) + \varepsilon$ and therefore all the points are out of the range $(f(a) - \varepsilon, f(a) + \varepsilon)$. Then the contradiction comes. And therefore $f(x)$ is bounded.

(b) **The Extreme Value Theorem**

Theorem 3. (Extreme Value Theorem) If $f(x)$ is continuous on $[\ell, r]$, then $f(x)$ has global extrema on $[\ell, r]$



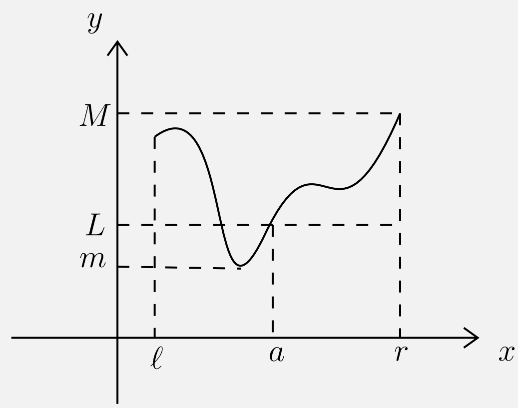
Proof. (Adapted from the notes by Dr. Fok-shuen Leung) By boundness theorem, we know $f(x)$ is bounded. We only have to prove that the $f(x)$ could reach its least upper bound and greatest lower bound. Without loss of generality, we assume that $f(x)$ does not attain its least upper bound U. Then we construct

$$g(x) = \frac{1}{U - f(x)}$$

which is a continuous function. By boundness theorem, we know $g(x) < V$ for some $V > 0$. Then $f(x) < U - \dfrac{1}{V}$ which contradicts to the fact that U is the least upper bound. Therefore, $f(x)$ could reaches it least upper bound and greatest lower bound. In other words, $f(x)$ has global extrema on $[\ell, r]$.

(c) **The Intermediate Value Theorem**

Theorem 4. (Intermediate Value Theorem) If $f(x)$ is continuous on $[\ell, r]$, then for any number L between $f(\ell)$ and $f(r)$, there exists a number $a \in [\ell, r]$ such that $f(a) = L$.



Proof. (Adapted from the assignments by Dr. Fok-shuen Leung) Without loss of generality, we assume $f(\ell) < f(r)$.

We divide $[\ell, r]$ into half and take the middle point to be m. If $f(m) = L$ then we are done. Otherwise Otherwise, if $f(m) > L$, let $[\ell, m]$ be the new interval, and if $f(m) < L$, let $[m, r]$ be the new interval. Name the new interval $[\ell_1, r_1]$.

Then divide $[\ell_1, r_1]$ in half, and call the midpoint m_1. If $f(m_1) = L$, we are done. Otherwise, if $f(m_1) > L$, let $[\ell_1, m_1]$ be the new interval, and if $f(m_1) < L$, let $[m_1 r_1]$ be the new interval. Name the new interval $[\ell_2, r_2]$.

Repeat the process ad infinitum,, we would finally get a sequence

$$[\ell_1, r_1] \supset [\ell_2, r_2] \supset [\ell_3, r_3] \supset \cdots$$

Since $\ell_n \leq \ell_{n+1} < r$ (increasing and bounded), we know $\{\ell_n\}$ converges. Let $\lim\limits_{n\to\infty} \ell_n = c \in [\ell, r]$. By continuity, we have

$$f(c) = \lim_{x\to c} f(x) = \lim_{n\to\infty} f(\ell_n) \leq L$$

Similarly, we know $\{r_n\}$ converges and let $\lim\limits_{n\to\infty} r_n = d \in [\ell, r]$. By continuity, we know

$$f(d) = \lim_{x\to d} f(x) = \lim_{n\to\infty} f(r_n) \geq L$$

Since we have

$$\lim_{n\to\infty}(r_n - \ell_n) = \lim_{n\to\infty}\frac{r-\ell}{2^n} = 0$$

we know

$$d - c = \lim_{n\to\infty} r_n - \lim_{n\to\infty} \ell_n = \lim_{n\to\infty}(r_n - \ell_n) = 0$$

This shows that we find a point $c = d$ such that

$$f(c) = L = f(d)$$

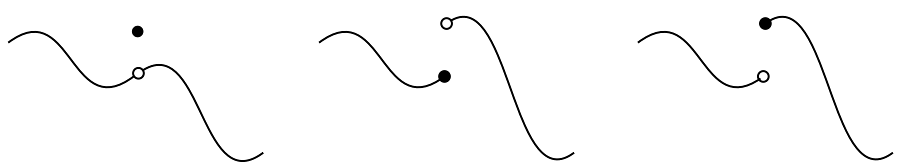
3. **Discontinuity**

After exploring the continuity, it may be helpful to explore the discontinuity. In other words, we want to explore how the rule

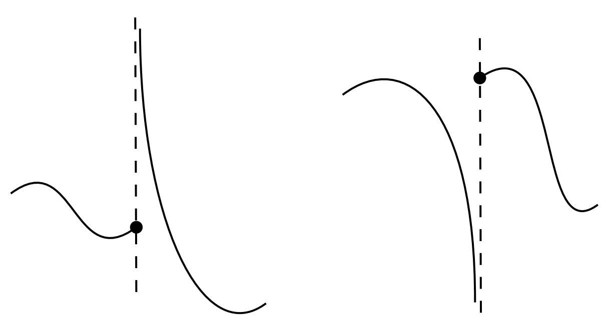
$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$$

is violated with the assumption that $x = a$ is in the domain of the function. We divide the violation into two cases.

- Case 1: Both limits exists but the equivalence is violated. $\lim_{x\to a^+} f(x) \neq f(a)$, $\lim_{x\to a^-} f(x) \neq f(a)$ or $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$. In other words, there are jumps or holes.



- Case 2: One of the limit does not exist. In other word, there is an asymptote.



In both cases, the function could be not bounded and the required value could be unreachable. Therefore, the boundness theorem, the extreme value theorem and the intermediate value theorem could not work on the discontinuity cases.

Problem Solving II

Determine the continuity; Apply the theorems of continuity

Problem 3: Determine the continuity

1. Determine the continuity of the function $f(x) = \dfrac{x^2-1}{x-1}$.
2. Determine the continuity of the function $f(x) = \begin{cases} 1, \, x \in \mathbb{Q} \\ 0, \, x \in \mathbb{R}/\mathbb{Q} \end{cases}$.
3. Determine the value of a such that the function $f(x) = \begin{cases} \dfrac{x^2+x-6}{x+3}, \, x \neq -3 \\ a, \, x = -3 \end{cases}$ is continuous.

Problem 4: Apply the theorems of the continuity

1. Calculate complicated limits
   (a) $\lim_{x\to 5} \dfrac{x^2 - \ln x}{\cos \pi x}$
   (b) $\lim_{x\to 5} \dfrac{\sin(\cos(\ln x))}{e^{\cos^2(\ln x)}}$
   (c) $\lim_{x\to e} \dfrac{\ln(\cos(\ln x^5))}{x^2}$

2. Find roots
   (a) Prove that if $f(x)$ is continuous on $[a, b]$ and $f(a)f(b) < 0$, there is more than one root in (a, b) .
   (b) Prove that $f(x) = x^3 - 15x + 1$ has at three roots on the interval $[-4, 4]$.

3. Prove that at a given instant there exists at least one pair of antipodal points on the earth which have the same temperature and same pressure. Assume the temperature and pressure distribution functions are continuous.

4. Let $f(x)$ be continuous and $0 \leq f(x) \leq 1$ on $[0, 1]$. Prove there exists $a \in [0, 1]$ such that $f(a) = a$.

5. Prove that if $f(x)$ is continuous on $[\ell, r]$ and the global maximum is M and the global minimum is m, then for any value $L \in [m, M]$ we can find a point $a \in [\ell, r]$ such that $f(a) = L$.

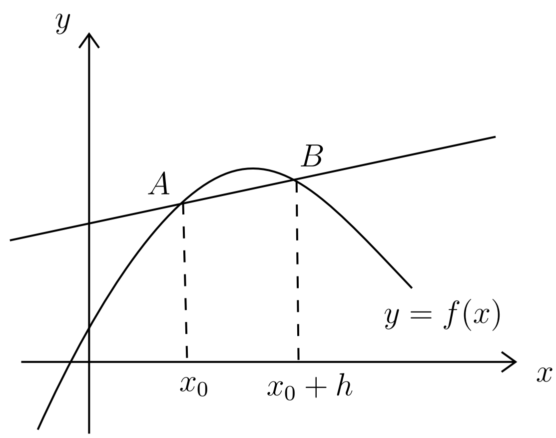
6. Let $f(x)$ be continuous on $[0, 2]$ and $f(0) + f(1) + f(2) = 3$. Prove there exists $a \in [0, 2]$ such that $f(a) = 1$.

7. Let $f(x)$ be continuous on $[\ell, r]$. Prove for any $p > 0$ and $q > 0$ there exists a point $a \in [\ell, r]$ such that $pf(\ell) + qf(r) = (p + q)f(a)$.

Derivatives

1. **Definition**

   Let's back to the big question in this course, what is the tangent line of the curve $y = f(x)$ at point $x = x_0$. In Latin, "tangent" means "to just touch". In other words, there is only one cross point between the straight line and the curve. To find it, we first let the line cross two points $A : (x_0, f(x_{,0}))$ and $B : (x_0 + h, f(x_0 + h))$.

   To get one point, we want to push the point B approaching to A, i.e. take the limit $h \to 0$. Then we could find out the slope of the tangent line

   $$k_{\text{slope}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

   We want to define the derivative to be the slope of the tangent line.

   Definition 7. The *derivative* of $f(x)$ is a function

   $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

   where the limit exists. And $f(x)$ is *differentiable* where $f'(x)$ or the limit exists.

   Then we want to explore the existence of the limit at certain point $x = x_0$. Since $h \to 0$ has two ways of approaching: $h \to 0^+$ and $h \to 0^-$, we separate the limit into two parts: the left derivative (left limit)

   $$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = f'_-(x_0)$$

   and the right derivative (right limit)

   $$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = f'_+(x_0)$$

For the limit to exist, we need the left limit and the right limit exist and equal to each other. Let's look at two examples.

Example 3. Consider $f(x) = x^2$, prove $f'(x) = 2x$.
By definition, we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x$$

Example 4. Consider $y = |x|$, prove it is not differentiable at $x = 0$.
By definition, we have

$$f'(0) = \lim_{h \to 0} \frac{|0+h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

Since

$$\frac{|h|}{h} = \begin{cases} 1, \ h > 0 \\ -1, \ h < 0 \end{cases}$$

we have

$$\lim_{h \to 0^+} = \frac{|h|}{h} = 1$$

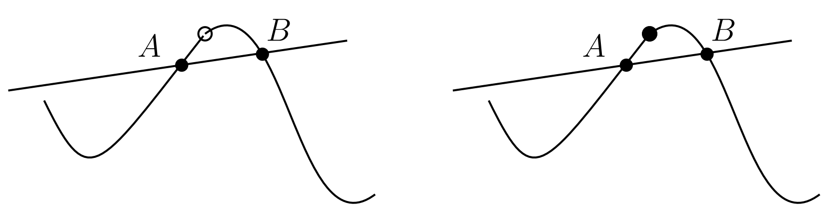
but

$$\lim_{h \to 0^-} = \frac{|h|}{h} = -1$$

Therefore, the limit does not exist and the derivative does not exist.

Note: For simplicity, we note that the first derivative is $f'(x) = \dfrac{df(x)}{dx}$, the second derivative is $f''(x) = \dfrac{d^2 f(x)}{dx^2}$ and the nth derivative is $f^{(n)}(x) = \dfrac{d^n f(x)}{dx^n}$.

2. **Differentiablity and Continuity**
Intuitively, if a function is differentiable, then, by definition, when we move point B approaching A, there should be no hole or jump - $f(x)$ should be continuous.



Bad Good

We could claim that the differentiablity implies the continuity.

> **Claim.** If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.

Proof. By definition, we have

$$\begin{aligned}\lim_{h\to 0}(f(a+h)-f(a)) &= \lim_{h\to 0} h\frac{f(a+h)-f(a)}{h}\\ &= \lim_{h\to 0} h \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\\ &= 0f'(a)=0\end{aligned}$$

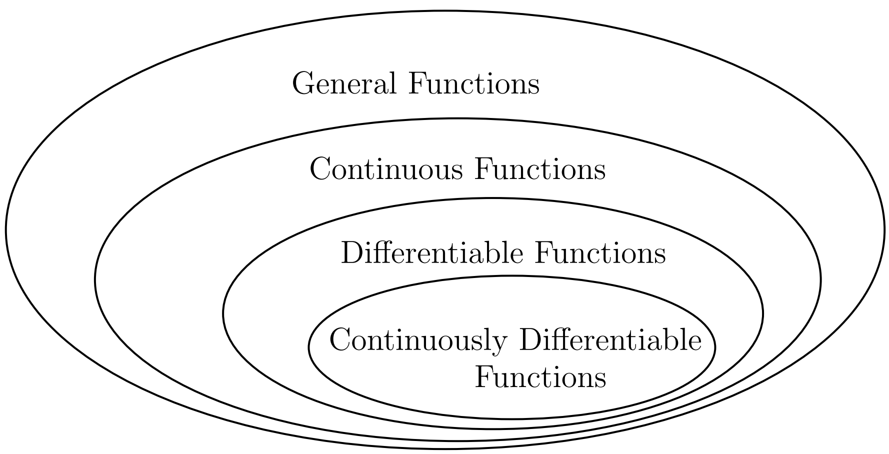
Therefore equivalently $\lim_{x\to a} f(x) = f(a)$, i.e. $f(a)$ is continuous at a.

Inspired by the discussion of differentiablity and continuity, we want to consider the case where the derivative (as a function) is continuous. Here we give the definition of continuously differentiable.

Definition 8. If the derivative of function $f(x)$ is continuous at $x = a$, then $f(x)$ is *continuously differentiable* at $x = a$

Note: The derivative of a (differentiable) function could be discontinuous but could only fall in case 2, i.e. there should be no jumps or holes. In summary we could find that

continuous differentiablity implies differentiablity, differentiablity implies continuity.



3. **Derivative Rules**
In practice, it is scary to calculate the derivative by definition. We wish to find some strategies to help us decompose those complicated functions such as $y = 6x^5 + \sin x$, $y = x \sin x$, $y = \dfrac{\ln x}{x}$ and $y = \log(\cos x)$. With such motivation, we explore five derivative rules as our tools.

In the following discussion, we assume $f(x)$ and $g(x)$ are differentiable.

(a) **Constant Multiplication Rule:** $(kf(x))' = kf'(x)$.
Proof. By definition, we have

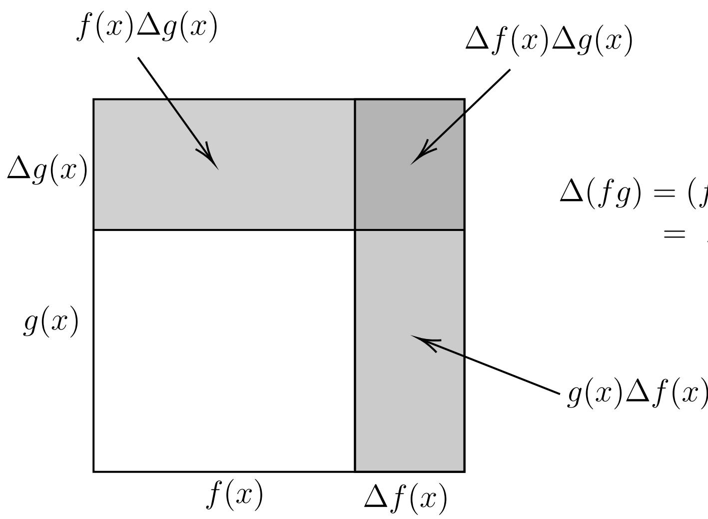
$$(kf(x))' = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} = k \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = kf'(x)$$

(b) **Sum Rule:** $(f(x) + g(x))' = f'(x) + g'(x)$
Proof. By definition, we have

$$\begin{aligned}(f(x) + g(x))' &= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x)\end{aligned}$$

Corollary: $(a_1 f_1 + \cdots + a_n f_n)' = a_1 f_1' + \cdots + a_n f_n'$

(c) **Product Rule:** $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
This could be understood with geometry. Consider the change of $f(x)$ and $g(x)$ with the change from x to $x + \Delta x$.



$$\begin{aligned}\Delta(fg) &= (f + \Delta f)(g + \Delta g) - fg \\ &= \Delta f g + f\Delta g + \underbrace{\Delta f \Delta g}_{\text{too small}}\end{aligned}$$

Then with intuition we take

$$\frac{\Delta(f(x)g(x))}{\Delta x} = \frac{\Delta f(x)}{\Delta x} g(x) + f(x)\frac{\Delta g(x)}{\Delta x} + \frac{\Delta f(x)\Delta g(x)}{\Delta x}$$

When we take $\Delta \to 0$, we have $\dfrac{\Delta f(x)\Delta g(x)}{\Delta x} \to 0$ by intuition. Then we have

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Then we could look at the formal proof of this rule.
Proof. By definition, we have

$$\begin{aligned}(f(x)g(x))' &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0} g(x+h)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0} f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)g(x)+f(x)g'(x)\end{aligned}$$

Corollary:

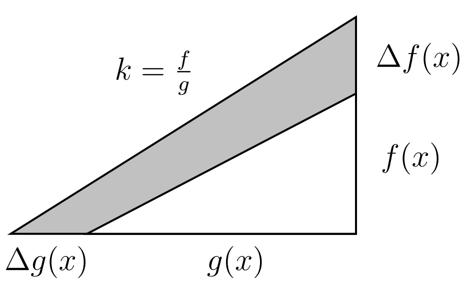
- $(f_1f_2\cdots f_n)' = (f_1'f_2\cdots f_n)+(f_1f_2'\cdots f_n)\cdots+(f_1f_2\cdots f_n')$
- $(fg)^{(n)} = \binom{0}{n}f^{(n)}g+\binom{1}{n}f^{(n-1)}g'+\cdots+\binom{n}{n}fg^{(n)}$

Alternatively, they could be represented as:

- $\left(\prod_{i=1}^{n} f_i(x)\right)' = \sum_{i=1}^{n} f_i'(x)f_1(x)\cdots\hat{f}_i(x)\cdots f_n(x)$
- $(fg)^{(n)} = \sum_{i=0}^{n}\binom{i}{n}f^{(n-i)}g^{(i)}$

(d) **Quotient Rule:** $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$

Again, we could look at its geometric interpretation. Consider the change of $f(x)$ and $g(x)$ with the change from x to $x+\Delta x$.



$$\begin{aligned}\Delta\left(\frac{f}{g}\right) &= \frac{f+\Delta f}{g+\Delta g}-\frac{f}{g}\\ &= \frac{(f+\Delta f)g-(g+\Delta g)f}{g(g+\Delta g)}\\ &= \frac{g\Delta f-f\Delta g}{g(g+\Delta g)}\end{aligned}$$

Now let's look at the formal proof.
Proof. By product rule, we know

$$\left(\frac{f(x)}{g(x)}\right) = f(x)\left(\frac{1}{g(x)}\right)'+f'(x)\left(\frac{1}{g(x)}\right)$$

By definition, we have

$$\begin{aligned}\left(\frac{1}{g(x)}\right)' &= \lim_{h\to 0}\left(\frac{1}{g(x+h)}-\frac{1}{g(x)}\right) = \lim_{h\to 0}\frac{g(x)-g(x+h)}{hg(x)g(x+h)}\\ &= -\lim_{h\to 0}\frac{1}{g(x)g(x+h)}\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = -\frac{g'(x)}{g^2(x)}\end{aligned}$$

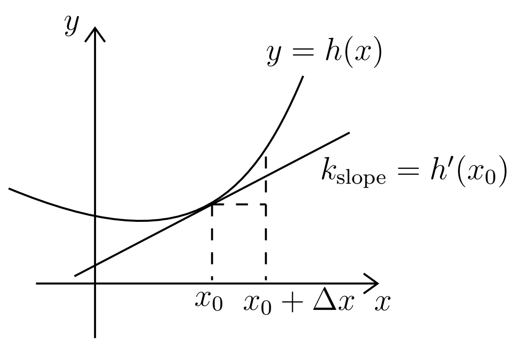
Hence, we could plug it in to get

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

(e) **Chain Rule:** $(f(g(x)))' = f'(g(x))g'(x)$
Intuitively, if we know the rate of u is related to v and the rate of u is related to x, we could have

$$\frac{\Delta u}{\Delta x} = \frac{\Delta u}{\Delta v}\frac{\Delta v}{\Delta x}$$

But we have to notice the case $\dfrac{\Delta h}{\Delta x} = \dfrac{\Delta h}{\Delta g}\dfrac{\Delta g}{\Delta x}$ where $\Delta g = 0$, i.e. $g(x+h) - g(x) = 0$. For example, $g(x) = x^2 \sin\left(\dfrac{1}{x}\right)$. Therefore, instead of division, we could use multiplication. We could let $h(x) = f(g(x))$ and make a linear approximation $h(x + \Delta x) \approx h(x) + h'(x)\Delta x$ (alternatively, we know $\Delta h(x) \approx h'(x)\Delta x$).



Linear Approximation

With the same linear approximation of $f(x)$ and $g(x)$, we know

$$\begin{aligned} h(x + \Delta x) &= f(g(x + \Delta x)) \\ &\approx f(g(x) + \underbrace{g'(x)\Delta x}_{\Delta g(x)}) \\ &\approx f(g(x)) + f'(g(x))g'(x)\Delta x \end{aligned}$$

Compare with the linear approximation of $h(x)$, we know

$$(f(g(x)))' = h'(x) = f'(g(x))g'(x)$$

With the same idea, we could give out the formal proof.

Proof. Construct a function

$$E(H) = \begin{cases} \dfrac{f(g(x)+H) - f(g(x))}{H} - f'(g(x)) & , H \neq 0 \\ 0 & , H = 0 \end{cases}$$

which is equivalent to

$$f(g(x)+H) - f(g(x)) = [f'(g(x)) + H]H \tag{*}$$

Let $H = g(x+h) - g(x)$, then (*) yields

$$f(g(x+h)) - f(g(x)) = [f'(g(x)) + E(g(x+h) - g(x))](g(x+h) - g(x))$$

Then by definition, we have

$$\begin{aligned} h'(x) &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} (f'(g(x)) + E(g(x+h) - g(x))) \\ &= \lim_{h\to 0} f'(g(x))g'(x) \end{aligned}$$

This proves

$$(f(g(x)))' = f'(g(x))g'(x)$$

With the help of chain rule, we could compute the implicit derivatives.

Example 5. Consider the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, find out the slope of the tangent line at $\left(2, \frac{3\sqrt{3}}{2}\right)$.

Take the derivative on both sides with repsect to x,

$$\frac{x}{8} + \frac{2}{9} y \frac{dy}{dx} = 0$$

which yields

$$\frac{dy}{dx} = -\frac{9x}{16y}$$

Then the slope would be

$$k = \left.\frac{dy}{dx}\right|_{x=2} = -\frac{9\times 2}{16\times \frac{3\sqrt{3}}{2}} = -\frac{\sqrt{3}}{4}$$

4. **Derivatives of Fundamental Functions**

   Based on the definition and derivatives rules, we could calculate the derivatives of some basic functions.

   (a) $\frac{d}{dx}(x^n) = nx^{n-1}\ (x \in \mathbb{N})$

Proof. By definition, we have

$$\begin{aligned}(x^n)' &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{x^n + nx^{n-1}h + \overbrace{P(x,h)}^{\substack{\text{only has } h^2 \text{ and}\\ \text{higher terms}}} - x^n}{h}\\ &= \lim_{h\to 0}\frac{nx^{n-1}h + P(x,h)}{h} = nx^{n-1}\end{aligned}$$

(b) $\dfrac{d}{dx}(x^{\frac{m}{n}}) = \dfrac{n}{m}x^{\frac{n}{m}-1}$ $(n, m \in \mathbb{N})$

Proof. The function is equivalent to $y^m = x^n$. Take the derivative on both sides with respect to x,

$$my^{m-1}y' = nx^{n-1}$$

Therefore we have

$$y' = \frac{n}{m}\frac{x^{n-1}}{x^{\frac{n}{m}(m-1)}} = \frac{n}{m}x^{\frac{n}{m}-1}$$

(c) $\dfrac{d}{dx}(x^{-n}) = -nx^{-n-1}$ $(n \in \mathbb{N})$

Proof. By definition, we have

$$\begin{aligned}\left(\frac{1}{x^n}\right) &= \lim_{h\to h}\left(\frac{\frac{1}{(x+h)^n} - \frac{1}{x^n}}{h}\right)\\ &= \lim_{h\to 0}\frac{x^n - (x+h)^n}{hx^n(x+h)^n} = -\lim_{h\to 0}\frac{1}{x^n(x+h)^n}\lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\ &= -\frac{1}{x^{2n}}nx^{n-1} = -nx^{-n-1}\end{aligned}$$

(d) $\dfrac{d}{dx}(e^x) = e^x$

Proof. e is defined as the number (around 2.718) such that the derivative of e^x evaluated at 0 is equal to 1. Then by definition

$$(e^x)' = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} = e^x\lim_{h\to 0}\frac{e^h - e^0}{h} = e^x f'(0) = e^x$$

(e) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

Proof. $y = \ln x$ is defined to be the inverse function of $y = e^x$, i.e. $e^y = x$. Take the derivative on both sides respect to x,

$$e^y \frac{dy}{dx} = 1$$

Then

$$y' = \frac{1}{e^y} = \frac{1}{x}$$

(f) $\dfrac{d}{dx}(a^x) = a^x \ln a$ ($a > 0$ and $a \neq 1$)

Proof. $\dfrac{d}{dx}(a^x) = \dfrac{d}{dx}\left(e^{x \ln a}\right) = e^{x\ln a} \ln a = a^x \ln a$

(g) $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$ ($a > 0$ and $a \neq 1$)

Proof. $\dfrac{d}{dx}(\log_a x) = \dfrac{d}{dx}\left(\dfrac{\ln x}{\ln a}\right) = \dfrac{1}{x \ln a}$

(h) $\dfrac{d}{dx}(\sin x) = \cos x$

Proof. By definition, we have

$$\begin{aligned}(\sin x)' &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{\overbrace{2\cos\frac{2x+h}{2}\sin\frac{h}{2}}^{\text{check Sum and Product Formulae}}}{h} \\ &= \lim_{h\to 0} \cos\left(x + \frac{h}{2}\right) \underbrace{\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}}_{=1} = \cos x\end{aligned}$$

(i) $\dfrac{d}{dx}(\cos x) = -\sin x$

Proof. By definition, we have

$$\begin{aligned}(\cos x)' &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} = \lim_{h\to 0} \frac{\overbrace{-2\sin\frac{2x+h}{2}\sin\frac{h}{2}}^{\text{check Sum and Product Formulae}}}{h} \\ &= -\lim_{h\to 0} \sin\left(x + \frac{h}{2}\right) \underbrace{\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}}_{=1} = -\sin x\end{aligned}$$

(j) $\tan x = \dfrac{1}{\cos^2 x}$

Proof. $(\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \dfrac{1}{\cos^2 x}$

(k) $\cot x = \dfrac{1}{\cos^2 x}$

Proof. $(\cot x)' = \left(\dfrac{\cos x}{\sin x}\right)' = \dfrac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = \dfrac{1}{\sin^2 x}$

(l) $\sec x = \sec x \tan x$

Proof. $(\sec x)' = \left(\dfrac{1}{\cos x}\right)' = -\dfrac{-\sin x}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \sec x \tan x$

(m) $\csc x = \sec x \tan x$

Proof. $(\csc x)' = \left(\dfrac{1}{\sin x}\right)' = -\dfrac{\cos x}{\sin^2 x} = -\dfrac{\cos x}{\sin^2 x} = -\csc x \cot x$

(n) $\dfrac{d}{dx}(x^a) = ax^{a-1}$ $(a \in \mathbb{R})$

Proof. Take logarithm on both sides,

$$\ln y = a \ln x$$

Take the derivatives on both sides with respect to x,

$$\frac{1}{y}y' = a\frac{1}{x}$$

Then

$$y' = a\frac{x^a}{x} = ax^{a-1}$$

(o) $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}$

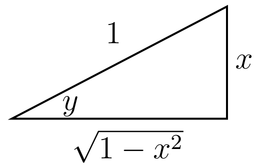
Proof. Equivalently, $y = \arcsin x$ is

$$\sin y = x$$

Take the derivatives with respect to x on both sides, we have

$$\cos y y' = 1$$

With the help of the triangle



We have

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

(p) $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}$

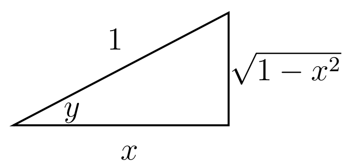
Proof. Equivalently, $y = \arccos x$ is

$$\cos y = x$$

Take the derivatives with respect to x on both sides, we have

$$-\sin y y' = 1$$

With the help of the triangle



We have

$$y' = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}$$

(q) $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$

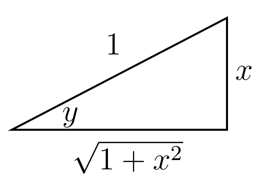
Proof. Equivalently, $y = \arcsin x$ is

$$\tan y = x$$

Take the derivatives with respect to x on both sides, we have

$$\frac{y'}{\cos^2 y} = 1$$

With the help of the triangle



We have

$$y' = \cos^2 y = \frac{1}{1+x^2}$$

In summary, we have

$f(x)$	$f'(x)$
x^a $(a \in \mathbb{R})$	ax^{a-1}
a^x $(a > 0 \text{ and } a \neq 1)$	$a^x \ln a$
$\log_a x$ $(a > 0 \text{ and } a \neq 1)$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\cot x$	$-\frac{1}{\sin^2 x}$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\cot x \csc x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$

Problem Solving III

Determine differentiablity and calculate the derivative by definition; Calculate the derivative with some basic application.

Problem 5: Determine differentiablity; Calculate the derivative by definition

1. Determine the differentiablity and calculate the derivative of $f(x) = \dfrac{1}{x}$ by definition.
2. Determine the differentiablity and calculate the derivative of $f(x) = |x^3|$ by definition.
3. Prove: A rational function is differentiable everywhere on its (maximal) domain. A *rational* function is $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Problem 6: Calculate the derivative

- **Complicated Derivatives**
  1. Calculate $\frac{d}{dx}\left(\frac{x^5+x^4-x+e^x}{x^2+x+\sin x}\right)$
  2. Calculate $\frac{d}{dx}\left(x^2\sin\left(\frac{1}{x}\right)+2x\right)$ $(x \neq 0)$
  3. Calculate $\frac{d}{dx}(\ln(\ln(\ln x)))$
  4. Find a tangent line of $f(x) = (e^x - 1)\sin x$ at $x = \pi$.

- **Higher Order Derivatives**
  5. Consider $f(x) = \frac{1}{ax+b}$.
     (a) Calculate $f'(x)$.
     (b) Calculate $f''(x)$.
     (c) Calculate $f^{(3)}(x)$.
     (d) Guess and prove $f^{(n)}(x)$.
  6. Prove: $(fg)^{(n)} = \sum_{i=0}^{n} \binom{i}{n} f^{(n-i)}g^{(i)}$ given $f(x)$ and $g(x)$ are differentiable.

- **Implicit Derivatives with Application**
  7. A 15 foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of 0.25 ft/sec. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing?
  8. Consider the function with the form $f(x) = u(x)^{v(x)}$.
     (a) Compute $\frac{d}{dx}x^x$ in the following two ways:
        i. Change the form that $f(x) = e^{x\ln x}$ and take the derivative.
        ii. Take the logarithm on both sides and take the derivative.
     (b) Compute $\frac{d}{dx}\left(u(x)^{v(x)}\right)$ given $u(x)$ and $v(x)$ are differentiable.
  9. Consider the *lemniscate of Bernoulli* $(x^2+y^2)^2 = 2a^2(x^2-y^2)$
     (a) Find out the tangent line at $(\sqrt{2}a, 0)$.
     (b) Sketch the lemniscate of Bernoulli.

Mean Value Theorem

1. **Background**

   Let's discuss the possible value of the derivative of $f(x)$ and try to interpret them. According to the definition of derivative, we could find out there are four cases: $f'(x) > 0$, $f'(x) < 0$, $f'(x) = 0$ and $f'(x)$ does not exist.

   Let's first discuss the **first two cases**: (I) $f'(x) > 0$ and (II) $f'(x) < 0$. Consider $f'(x_0) > 0$, based on the definition, we know

   $$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} > 0$$

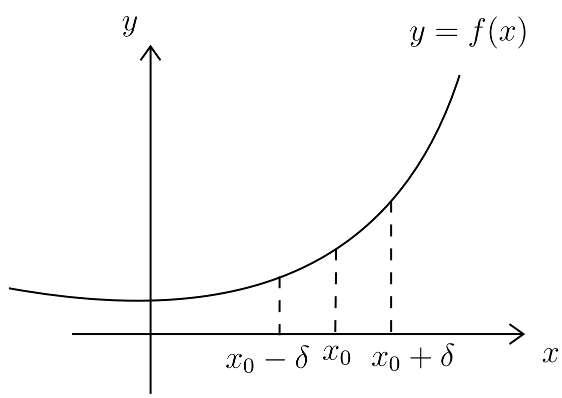
   which means when $0 < h < |\delta|$,

   $$\frac{f(x_0 + h) - f(x_0)}{h} > 0$$

   Therefore, we know in a small interval of h

   $$\begin{cases} f(x_0 + h) > f(x_0) & \text{if } h > 0 \\ f(x_0 + h) < f(x_0) & \text{if } h < 0 \end{cases}$$

   This means in around point x_0, the value of the right interval is higher than $f(x_0)$ while the left interval is lower than $f(x_0)$.

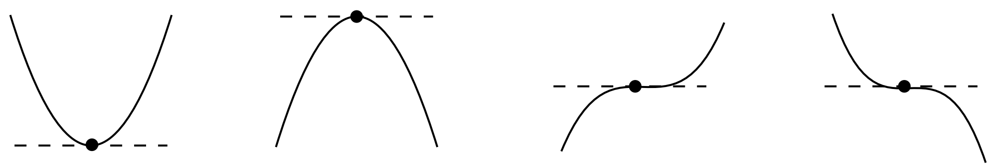
   

   If we have an interval (ℓ, r) such that when $x \in (\ell, r)$,$f'(x) > 0$, we could know any point $x \in (\ell, r)$ is lower than the points to the right of this point. It is very likely that if $\ell < x_1 < x_2 < r$, $f(x_1) < f(x_2)$. We want to call this case as increasing.
   Similarly, if we have an interval (ℓ, r) such that when $x \in (\ell, r)$,$f'(x) < 0$, we could know any point $x \in (\ell, r)$ is higher than the points to the right of this point. It is very likely that if $\ell < x_1 < x_2 < r$, $f(x_1) > f(x_2)$. We want to call this case decreasing.
   Then we could form the definition of increasing and decreasing functions.

Definition 9. $f(x)$ is increasing on (ℓ, r) if for any $\ell < x_1 < x_2 < r$, $f(x_1) < f(x_2)$. $f(x)$ is decreasing on (ℓ, r) if for any $\ell < x_1 < x_2 < r$, $f(x_1) > f(x_2)$.

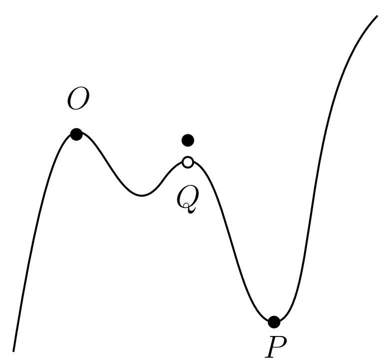
However, whether $f'(x) > 0(< 0)$ guarantee the monotonicity (increasing/decreasing) requires the further proof.

Then let's discuss the **rest two cases**: (III) $f'(x) = 0$ and (IV) $f'(x)$ does not exist. Consider $f'(x_0) = 0$, there are four possible cases: f(x) remains unchanged at x_0 and could be increase or decrease on both sides of the neighborhood with continuity.



A B C D

Consider case A and B, $f(x_0)$ seems to be the maximum or minimum. However, this is not always the case. Fox example,



Point O and point P are locally maximum and minimum in their small neighbourhood but not the (global) maximum or minimum in the entire interval. Then want to construct a definition on this kind of situation.

Definition 10. Let $y = f(x)$, $x \in D$ and $x_0 \in D$. If there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$, $f(x) \leq f(x_0)$, $f(x)$ has a *local maximum* at x_0. If there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$, $f(x) \geq f(x_0)$, $f(x)$ has a *local minimum* at x_0.

Then here comes a question: where does the local extremum exist based on the value of $f'(x)$? It is clear that case (I) and case (II) does not have local extremum as we have shown. Then the possibilities lie on case (III) and (IV). We have already shown that case (III) could have local extremum. If we check the example above, point Q is a local maximum and $f'(x)$ does not exist at that point. We could conclude that case (IV) also have the local extremum. Therefore we know local extremum implies $f'(x) = 0$ or $f'(x)$ does not exists. This is the interior extremum theorem with the definition of critical point.

Definition 11. $f(x)$ has *critical point* at x_0 if $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

Theorem 5. (Interior Extremum Theorem) If $f(x)$ has a local extremum at x_0, then $f(x)$ has a critical point at x_0.

Proof. Without loss of generality, we assume $f(x)$ has local minimum at x_0. If $f'(x)$ does not exist then it is a critical point. If $f'(x)$ exists, we know

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

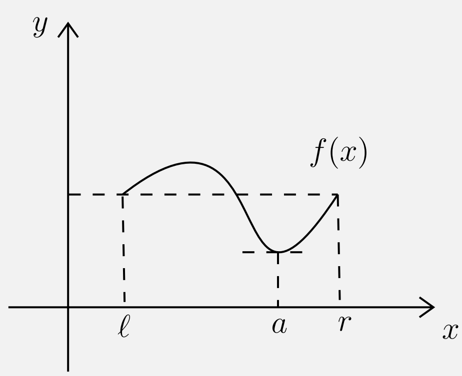
There exists $\delta > 0$ that $0 < |x| < \delta$, $\dfrac{f(x_0 + h) - f(x_0)}{h}$ has the same sign as $f'(x_0)$. Consider $h > 0$ we should have $f(x_0 + h) > f(x_0)$, which means $f'(x_0) \geq 0$. Consider $h < 0$, we should have $f'(x_0) \leq 0$. Therefore $f'(x_0) = 0$.

Note: critical points do not guarantee the existence of local extremum. Recall case (III C) and (III D), with $f'(x_0) = 0$, they are not local extremum. Besides a sharp change ($f'(x_0) \neq 0$) does not guarantee the existence of local extremum. For example, $f(x) = \begin{cases} 2x, \ x < 0 \\ 3x, \ x \geq 0 \end{cases}$ does not have a local extremum at $x = 0$.

2. **Rolle Theorem**

As a consequence of the interior extremum theorem and extremum value theorem, we have the Rolle theorem established.

Theorem 6. (Rolle Theorem) Let $f(x)$ be continuous on $[\ell, r]$ and differentiable on (ℓ, r), with $f(\ell) = f(r)$. Then there exists a number a in (ℓ, r) such that $f'(a) = 0$.



Proof. If $f(x)$ is a constant function, i.e. $f(x) = c$, then it is obvious that $f'(x) = 0$ for all $x \in [\ell, r]$. If $f(x)$ is not a constant function, then it is not possible for $f(\ell) = f(r)$ to be the global maximum and global minimum at the same time.Therefore, there should at least be one on global maximum and global minimum existing in (ℓ, r). By definition, global extremum should also be local extremum. Since $f(x)$ is differentiable on (ℓ, r), at that local extrema point $a \in (\ell, r)$, $f'(a) = 0$.

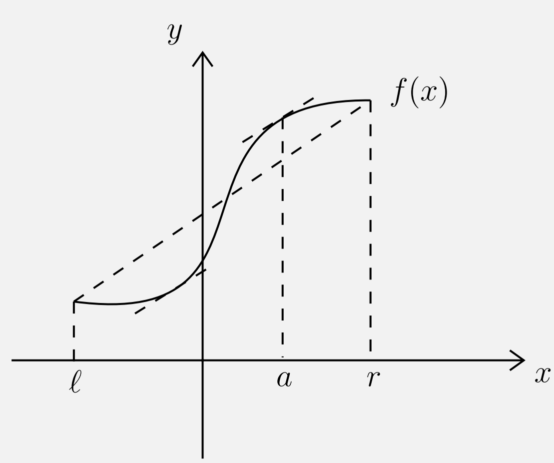
This is our first powerful tool to prove claims related derivatives.

Example 6. Let $f(x)$ to be continuous on $[a, b]$ and differentiable on (a, b). Let $f(a)f(b) > 0$ and $f(a)f\left(\frac{a+b}{2}\right) < 0$. Prove there exists a number $c \in (a, b)$ such that $f'(c) = 0$.

Solution. Since $f(x)$ is continuous and $f(a)f\left(\frac{a+b}{2}\right) < 0$, by intermediate value theorem, we know there exists $c_1 \in \left(a, \frac{a+b}{2}\right)$ such that $f(c_1) = 0$. And by $f(a)f(b) > 0$, we know $f(b)f\left(\frac{a+b}{2}\right) < 0$. Therefore, there exists $c_2 \in \left(\frac{a+b}{2}, b\right)$ such that $f(c_2) = 0$. Since $a < c_1 < c_2 < b$, we know $f(x)$ is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2). Then by Rolle theorem, there exists a number $c \in (c_1, c_2) \subset (a, b)$ such that $f'(c) = 0$.

3. **Mean Value Theorem**

Theorem 7. (Mean Value Theorem, by Lagrange) Let $f(x)$ be continuous on $[\ell, r]$ and differentiable on (ℓ, r). Then there exists a number a in (ℓ, r) such that $f'(a) = \frac{f(r) - f(\ell)}{r - \ell}$.



Mean values theorem shows that we can find a point with tangent line parallel to the line connected two endpoints. Or it could be interpreted as we can find a number a in between ℓ and r with $f'(a)$ to be the average (mean value) of the slope between two end-points $(\ell, f(\ell))$ and $(r, f(r))$. We could observe that mean value theorem is generalized Rolle theorem (with $f(\ell) = f(r)$). On the other hand, we could threat mean value theorem is shifted from Rolle theorem with a straight line $y = g(x) = \dfrac{f(r) - f(\ell)}{r - \ell}(x-r)+f(r)$.

Proof. To connect to Rolle theorem, we construct a function

$$\varphi(x) = f(x) - g(x) = f(x) - \frac{f(r) - f(\ell)}{r - \ell}(x - r) - f(r)$$

We could have $\varphi(\ell) = \varphi(r) = 0$. Since $\varphi(x)$ is continuous on $[\ell, r]$ and differentiable on (a, b), by Rolle theorem, there exists a number $a \in (\ell, r)$ such $\varphi'(a) = f'(x) - \dfrac{f(r) - f(\ell)}{r - \ell} = 0$. This means there exists a number $a \in (\ell, r)$ such that $f'(a) = \dfrac{f(r) - f(\ell)}{r - \ell}$.

This is our second powerful tool to prove claims related derivatives.

Corollary: If $f'(x) > 0$ on an interval D, then $f'(x)$ is increasing on D. If $f'(x) < 0$ on an interval D, then $f'(x)$ is decreasing on D.

Proof. Without loss of generality, we let $f'(x) > 0$ on D. Choose two points $x_1 < x_2$ and $x_1, x_2 \in D$, then $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Therefore, by mean value theorem, we can find a number $a \in (x_1, x_2)$ such that

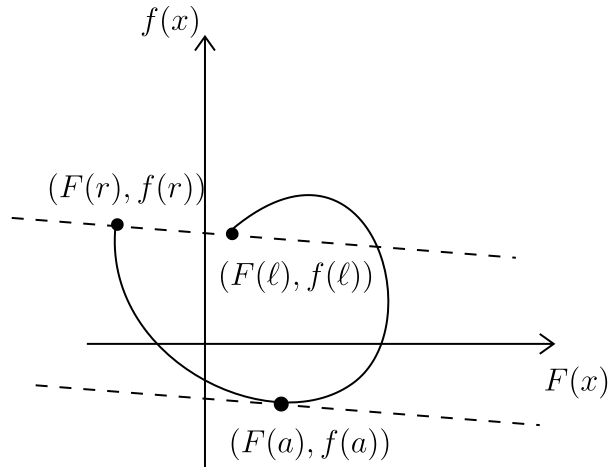
$$f(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

which implies $f(x_2) > f(x_1)$ and $f(x)$ is increasing.

4. **Generalized Mean Value Theorem**
We could continuously generalize the mean value theorem.

(Generalized Mean Value Theorem, by Cauchy) If $f(x)$ and $F(x)$ are continuous on $[\ell, r]$ and differentiable on (ℓ, r), and $F'(x) \neq 0$ on (ℓ, r), then there exists $a \in (\ell, r)$ such that $\dfrac{f(r) - f(\ell)}{F(r) - F(\ell)} = \dfrac{f'(a)}{F'(a)}$.

This could are visualized via a parameterized curve $\vec{r}(x) = (F(x), f(x))$ $(a \leq x \leq b)$. Generalized mean value theorem shows that you could find a point on the curve with the tangent line parallel to the straight line connecting two end points $(F(\ell), f(\ell))$ and $(F(r), f(\ell))$. In other words, we can find a number a between ell and r with $\dfrac{f'(a)}{F(a)}$ to be the average (mean value) of the slope between two end-points. If we let $F(x) = x$, it is just mean value theorem.



$$k = \lim_{\Delta F \to 0} \frac{\Delta f}{\Delta F} = \lim_{\Delta x \to 0} \frac{\frac{\Delta f}{\Delta x}}{\frac{\Delta F}{\Delta x}} = \frac{f'(a)}{F'(a)}$$

Proof. Inspired by the proof of mean value theorem, we want to construct a function $\varphi(x)$ to connect to the Rolle theorem. We let

$$\varphi(x) = f(x) - f(\ell) - \frac{f(r) - f(\ell)}{F(r) - F(\ell)}(F(x) - F(\ell))$$

where $\varphi(\ell) = \varphi(r) = 0$ and $\varphi(x)$ is continuous on $[\ell, r]$ and differentiable on (ℓ, r). Therefore, by Rolle theorem, we can find a number a between ℓ and r such that

$$\varphi'(a) = f'(a) - \frac{f(r) - f(\ell)}{F(r) - F(\ell)}F'(a) = 0$$

which can be simplified as $\dfrac{f(r) - f(\ell)}{F(r) - F(\ell)} = \dfrac{f'(a)}{F'(a)}$.

This is our third powerful tool to prove claims related to the derivatives.

Example 7. Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b) where $a > 0$. Prove there exists a number $c \in (a, b)$ such that $f(b) - f(a) = cf'(c) \ln \dfrac{b}{a}$.

Solution. The equation is equivalent to $\dfrac{f(b) - f(a)}{\ln b - \ln a} = \dfrac{f'(c)}{\frac{1}{c}}$. Compared to the equation in generalized mean value theorem, we could know $F(x)$ should be $\ln x$. Then we let $F(x) = \ln x$ which is continuous on $[a, b]$ and differentiable on (a, b). Therefore, by generalized mean value theorem, we know we can find a number $c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{\ln b - \ln a} = \dfrac{f'(c)}{\frac{1}{c}}$.

5. **L'Hospital's Rule**

When we calculate the limit, we may counter some indeterminate forms such as $\frac{0}{0}$ and $\frac{\infty}{\infty}$. As a consequence of generalized mean value theorem, L'Hospital's Rule is a strong tool to solve this kind of problem. Instead of comparing the number they are approaching, L'Hospital's rule compares the speed the are approaching to 0 or ∞.

Theorem 8. (L'Hospital's Rule) Let $f(x)$ and $g(x)$ be differentiable on an open interval I containing a number a. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\pm\infty$ and $g(x) \neq 0$ on the interval $I/\{a\}$. Then we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right-hand side exists or is ∞ or $-\infty$.

Example 8. First consider a $\frac{0}{0}$ indeterminate form

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$$

Then consider a $\frac{\infty}{\infty}$ indeterminate form

$$\lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0$$

Problem Solving IV

Application of Rolle theorem, mean value theorem, generalized mean value theorem and L'Hospital's rule.

Problem 6: Application of Rolle theorem, mean value theorem, generalized mean value theorem and L'Hospital's rule

- **Proof of Claims Related to Derivatives**

1. Let $f(x)$ to be continuous and differentiable on $[0,2]$. $3f(0) = f(1) + 2f(2)$. Prove: there exists a number $a \in (0,2)$ such that $f'(a) = 0$.
2. Let $f(x)$ to be continuous on $[0,1]$ and differentiable on $(0,1)$. $f(1) = 0$. Prove there exists a number $a \in (0,1)$ such that $af'(a) + 3f(a) = 0$.
3. Let $f(x)$ to be continuous on $[a,b]$ and differentiable on (a,b). Prove $f(b) - f(a) = \dfrac{f'(c)}{2c}(b^2 - a^2)$ where $a > 0$.
4. Let $f(x)$ and $g(x)$ to be continuous on $[a,b]$ and differentiable on (a,b). $g'(x) \neq 0$. Prove there exists a number $c \in (a,b)$ such that
$$\frac{f(b) - f(c)}{g(c) - g(a)} = \frac{f'(c)}{g'(c)}$$
5. Let $f(x)$ to be continuous on $[0,1]$ and differentiable on $(0,1)$. $f(0) = 0$ and $f(1) = 1$. Prove that there exists a number $c \in (0,1)$ such that $f(c) = 1 - c$ and there exists $a, b \in (0,1)$ such that $f'(a)f'(b) = 1$.
6. Let $f(x)$ to be continuous on $[a,b]$ and differentiable on (a,b) where $a > 0$. Prove there exists $c, d \in (a,b)$ such that $f'(c) = \dfrac{f'(d)}{2d}(a+b)$.

- **Proof of Inequality**

7. Prove when $x > 0$, $e^x - 1 > x$.
8. Prove when $x > 0$, $\dfrac{x}{1+x} < \ln(1+x) < x$

- **Application of L'Hospital's Rule: Evaluate Indeterminate Forms**

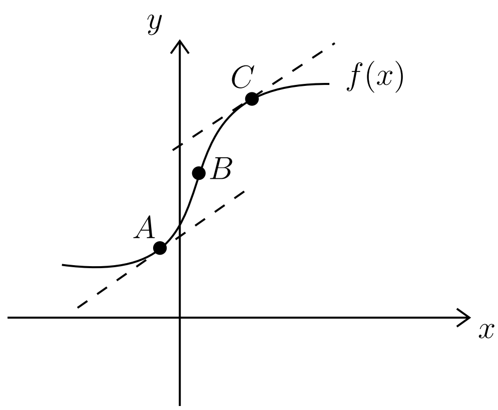
9. Evaluate $\lim\limits_{x\to\infty} \dfrac{x^n}{a^x}$ $(a > 1,\, n \in \mathbb{N}_+)$
10. Evaluate $\lim\limits_{x\to 0} \dfrac{\arctan 2x + e^{-x} - 1}{\sin x}$
11. Evaluate $\lim\limits_{x\to 0^+} x^x$
12. Evaluate $\lim\limits_{x\to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$

Application of Differential Calculus

1. **Curve Sketching**
   To sketch a curve $f(x)$ is to find the domain, the monotonicity with critical points, the concavity with inflection points, the (local) extremum, the intercepts and the asymptotes.

   (a) **Monotonicity**
   Recall from the previous section, we know that if $f'(x) > 0$ on an interval D, then $f'(x)$ is increasing on D and if $f'(x) < 0$ on an interval D, then $f'(x)$ is decreasing on D.

   (b) **Concavity**
   Basically, concavity describes how the function is curved with respect to a straight line. Consider the following example.



Intuitively we say at point A the is curve concave up and at point C the is curve concave down. Besides, the curve change its concavity at point B. As a reference, at point A the curve is above its tangent line and at point C the curve is below its tangent line. Therefore, we could construct the definition of concavity based on the tangent line.

Definition 12. We say the function $f(x)$ is *concave up* on an interval if in that in that interval the curve is above its tangent line. And we say the function $f(x)$ is *concave down* on an interval if in that in that interval the curve is below its tangent line. If $f(x)$ changes concavity at $x = a$, then we say $f(x)$ has an inflection point at $x = a$.

Let's mask some further observation. We could find when the function is concave up, the slope of its tangent line is increasing (possibly $f''(x) > 0$) and when the function is concave down, the slope of its tangent line is decreasing (possibly $f''(x) < 0$). Then an inflection point where the function changes the concavity should has $f''(x) = 0$ or $f''(x)$ does not exist.

Theorem 9. $f(x)$ is concave up on an interval I if $f''(x) > 0$ on that interval I. $f(x)$ is concave up on an interval I if $f''(x) < 0$ on that interval I.

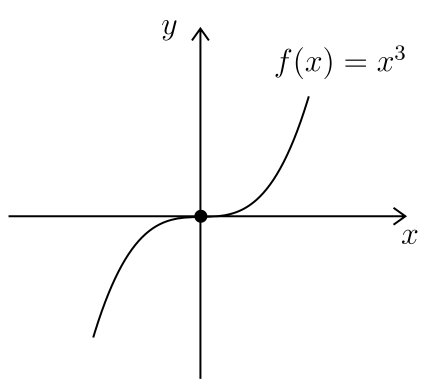
Proof. Without loss of generality, let $f'(x)$ increase on the interval I. Consider a point $x_0 \in I$ and the tangent line $L(x) = f'(x_0)(x - x_0) + f(x_0)$. Construct a distance function between the curve and the straight line $\varphi(x) = f(x) - L(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$. If $x > x_0$, by mean value theorem, we can find a number $c \in (x_0, x)$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(c) > f'(x_0)$$

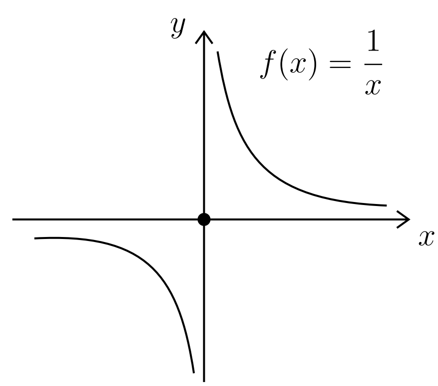
since $f'(x)$ is increasing ($f''(x) > 0$). Then $\varphi(x) = f(x) - f(x_0) - f'(x_0)(x - x_0) > 0$. With the same strategy, we could prove if $x < x_0$, $\varphi(x) = f(x) - f(x_0) - f'(x_0)(x - x_0) > 0$. In conclusion, the curve is above the its tangent line.

Claim. If $f(x)$ has an inflection point at $x = a$, then $f''(a) = 0$ or $f''(a)$ does not exist.

Proof. $f''(a)$ has four possibilities: $f''(a) > 0$, $f''(a) > 0$, $f''(a) > 0$ and $f''(a)$ does not exist. Since first two cases implies either the function is concave up or down, the possibilities for the change of concavity lie on $f''(a) > 0$ and $f''(a)$ does not exist. We only need to show there exists cases that when $f''(a) > 0$ and $f''(a)$ does not exist, $x = a$ is an inflection points. The following two examples fulfill our requirements ($x = 0$ is an inflection point).



$f''(0) = 0$



$f''(0)$ DNE

(c) **Local Extremum**

Recall from the previous section, the local extremum should be at a critical point but the critical point does guarantee the local extremum. Therefore, a direct way to find local extremum is first find out all critical points and then check whether the way local extremum (check the value of neighbor points).

Here we introduce another strategy called **first derivative test**. At a critical point

x_0, if $\begin{cases} f'(x) < 0, \ x < x_0 \\ f'(x) > 0, \ x > x_0 \end{cases}$ then x_0 is the local minimum; if $\begin{cases} f'(x) > 0, \ x < x_0 \\ f'(x) < 0, \ x > x_0 \end{cases}$ then x_0 is the local maximum.

(d) **Asymptote**

Recall from the first section, we have three types of asymptotes:

- If we have $\lim_{x \to a} f(x) = \pm\infty$ and , we have $x = a$ and $x = b$ to be the *vertical asymptotes*.
- If we have $\lim_{x \to \pm\infty} f(x) = a$, we have $y = a$ to be the *horizontal asymptote*.
- More generally, if we have $\lim_{x \to \infty} [f(x) - (kx + b)] = 0$, then we have $y = kx + b$ to the *oblique asymptote*.

In summary, we have the following receipt to sketch the curve:

1° Find out the domain of the function $x \in D$.
2° Find out $f'(x)$. Classify where $f'(x) > 0$ (increasing) and $f'(x) < 0$ (decreasing). Find out all the critical points and apply first derivative test on them.
3° Find out $f''(x)$. Classify where $f''(x) > 0$ (concave up) and $f''(x) < 0$ (concave down). Find out all the points with $f''(x) = 0$ or $f''(x)$ does not exist. And check whether these points are inflection points.
4° Find out all he intercepts.
5° Summarize all the information in a table
6° Find out all the asymptotes.
7° Sketch the curve.

Let's look at an example.

Example 9. Sketch $y = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

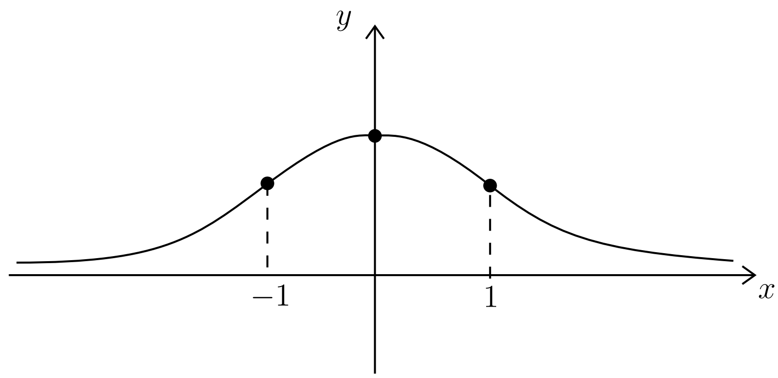
We know the domain is simply $\mathbb{R}$. Let $f'(x) = -\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = 0$, we have the critical point to be $x = 0$. Let $f''(x) = \frac{1}{\sqrt{2\pi}} (x^2 - 1) e^{-\frac{-x^2}{2}} = 0$, we have $x = \pm 1$. Then we could summarize those information in a table.

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$f'(x)$	$+$		$+$		$-$		$-$
$f''(x)$	$+$		$-$		$-$		$+$
$f(x)$	$\nearrow$ (up)	infle	$\nearrow$ (down)	lo max	$\searrow$ (down)	infle	$\searrow$ (up)

Then we calculate the limit

$$\lim_{x \to \infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \lim_{x \to -\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = 0$$

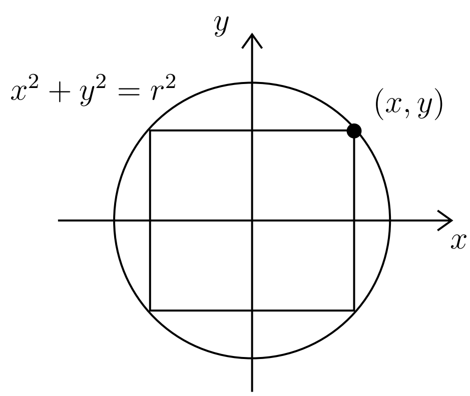
Therefore we have a horizontal asymptote $y = 0$. Then finally we could sketch the curve.



2. **Optimization**
   Basically, the key points of optimization are modelling of the function, finding out all the local extremum and finding out the global extremum. Here is a receipt.

   1° Build a model
   2° Come up with a objective function.
   (If necessary, reduce the objective function to one variable).
   3° Identify the domain.
   4° Optimize: find out all the local extremum and find out the global extremum (do not forget *endpoints* to compare with) based on the extremum value theorem and first derivative test.
   5° Apply reality check.

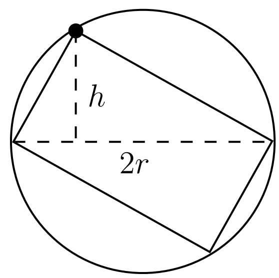
   Example 10. What is the rectangle of the largest area that can be inscribed in a circle of radius r?
   We know the graph is



   And the area of the rectangle is $A(x, y) = 4xy$ which could be reduced as $A(x) = 4x\sqrt{r^2 - x^2}$. And we know the domain should be $x \in [0, r]$. Then we could do the optimization. We have

$$A'(x) = 4\left(\sqrt{r^2 - x^2} - \frac{x}{\sqrt{r^2 - x^2}}\right)$$

Let $A'(x) = 0$ we have $x = \dfrac{\sqrt{2}}{2}r$. With the first derivative test, we know it is the local maximum. Then we check $A(0) = 0$, $A(r) = 0$ and $A\left(\dfrac{\sqrt{2}}{2}r\right) = 2r^2$. Then we have a global maximum at $x = \dfrac{\sqrt{2}}{2}r$. We could have a cross check here. Consider the rectangle is fixed at the diameter and the vertex is movable.



$$A = \frac{1}{2}(2r)h = rh$$

The area A is maximum when the height h is maximum. The height h is maximum when the rectangle is a square when $x = \dfrac{\sqrt{2}}{2}r$ in the coordinate system above.

Problem Solving V

Curve Sketching; Optimization

Problem 8: Curve Sketching: monotonicity, concavity, asymptote

1. Let's discuss the property of function $f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), \ x \neq 0 \\ 0, \ x = 0 \end{cases}$.

   (a) Sketch $f(x)$ $(x \neq 0)$.

   (b) Prove $\lim_{x \to 0} \sin\left(\dfrac{1}{x}\right)$ does not exist and then conclude $f(x)$ is not continuous at $x = 0$.

   (c) Consider $g(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right), \ x \neq 0 \\ 0, \ x = 0 \end{cases}$. Sketch $g(x)$.

   (d) Prove $g(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$.

   (e) Consider $h(x) = \begin{cases} x^2\sin\left(\dfrac{1}{x}\right), \ x \neq 0 \\ 0, \ x = 0 \end{cases}$. Sketch $h(x)$.

   (f) Prove $h(x)$ is differentiable at $x = 0$ but $h'(x)$ is not continuous at $x = 0$.

2. Let

$$f(x) = \begin{cases} x + 2x^2\sin\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   Use the definition of derivative to show that $f'(0) = 1$, but then show that $f(x)$ is not increasing on any interval $(-\delta, \delta)$.

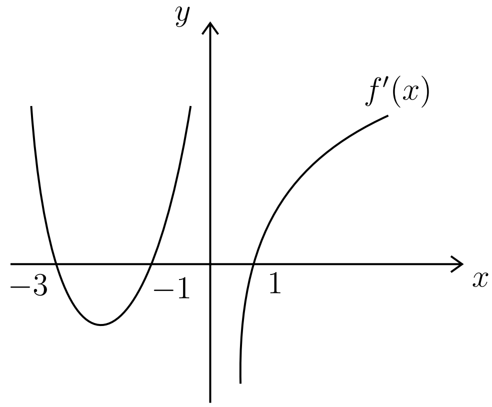
3. Find the region of decreasing/increasing of $f(x) = x^3 - x^2 + x + 1$.

4. Find the region of concave up/concave down of $f(x) = x^3 - x^2 + x + 1$.

5. Sketch $V(r) = \varepsilon\left(\left(\dfrac{R}{r}\right)^6 - 2\left(\dfrac{R}{r}\right)^{12}\right)$

Problem 9: Optimization: local extremum and global extremum

1. Find the local extremum of $f(x) = x^3 - 12x + 5$.
2. Find the global extremum of $f(x) = \ln x - \dfrac{x}{e} + 2$.
3. Find the global extremum of $f(x) = x^p + (1-x)^p$ where $p > 1$ and $0 \le x \le 1$.
4. Consider $f(x)$ with derivative shown in the graph



   Find the local extremum of $f(x)$.
5. Let $e < a < b$, Prove $a^b > b^a$.
6. Given a fixed surface area, what is the maximum volume of a cylindrical can of that surface area?
7. What is the area of the largest rectangle (with sides parallel to the axes) which may be inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$?
8. Suppose you wish to connect four points at the corners of a square. What is the total length of the shortest path? (The path may have branches.)
   (a) First prove the result with pre-calculus knowledge (basic geometry).
   (b) Then prove the result with differential calculus.