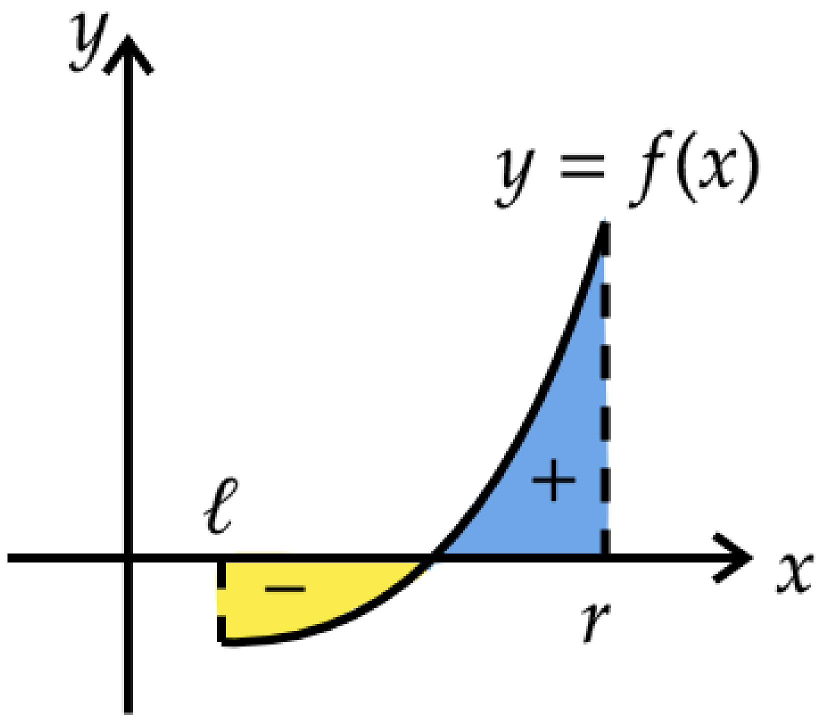


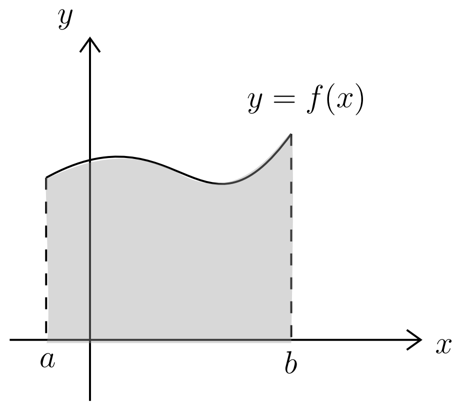
MATH 101: Integral Calculus with One Variable

Wucheng Zhang

http://blogs.ubc.ca/wucheng/

Introduction

The big question in this course to find the area under $y = f(x)$ $(a < x < b)$.

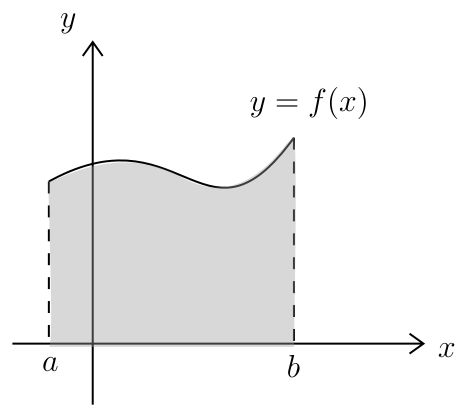


This requires us to have **INTEGRAL**. To evaluate it, we need the **FUNDAMENTAL THEOREM OF CALCULUS** and some **TECHNIQUES**. With these tools, we could apply the integral on **VOLUME**, **WORK** and **DIFFERENTIAL EQUATION**. Finally, with the combination of the knowledge of sequence, limit, derivative and integral, we are going to study **SERIES** and its **APPLICATION**.

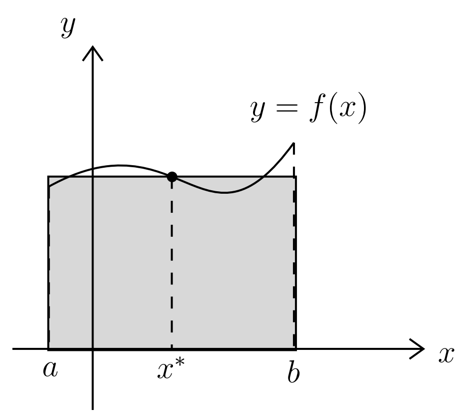
Contents

Integral and Fundamental Theorem of Calculus

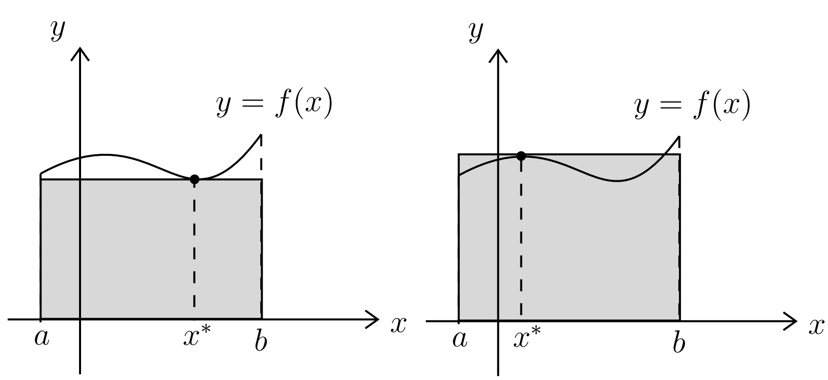
1. **Definition of Integral and Integrablity**
   The big question in this course to find the area under $y = f(x)$ $(a < x < b)$.



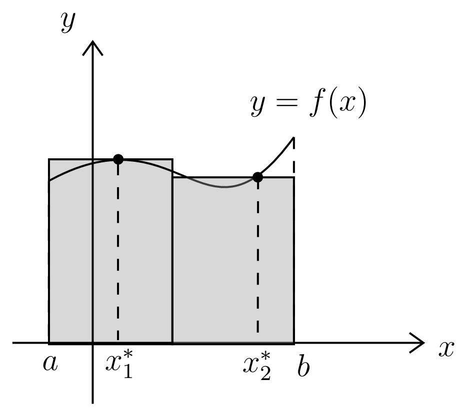
The first attempt is to make an approximation with a rectangle which is easy to calculate the area. Then we have



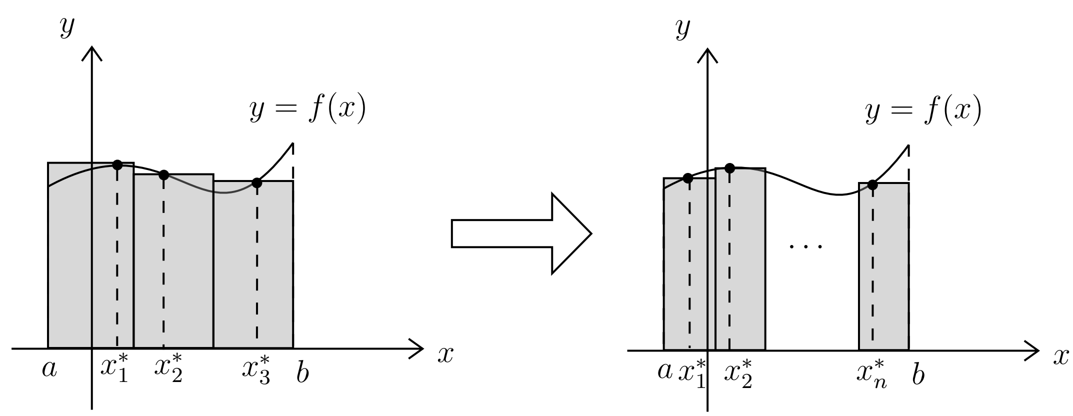
where the area is approximated to be $A \approx (b-a)f(x^*)$ and we call x^* to be the representative point who's value $f(x^*)$ represents the average height of the function. However, the accuracy is very low and there are various ways to choose the representative points which gives different results.



Therefore we want to look for more accurate approximation. A direct attempt is to divide the function into two pieces and make the approximation on each pieces.



Then the area is approximated to be $A \approx \dfrac{b-a}{2}(f(x_1^*) + f(x_2^*))$. We could find that the approximation is more accurate and the variation of representative points is smaller. Then we are motivated to divide the function (domain) into smaller pieces.



We could see that the more pieces we take, the more accurate the approximation is and also the smaller the variation of representative points is. With n pieces, the area is approximated to be $A \approx \dfrac{b-a}{n} \displaystyle\sum_{i=1}^{n} f(x_i^*)$. By intuition we want to take the $n \to \infty$ to get the most accurate result.

Then based on the above motivation, we could have the following of integral which calculates the area we need.

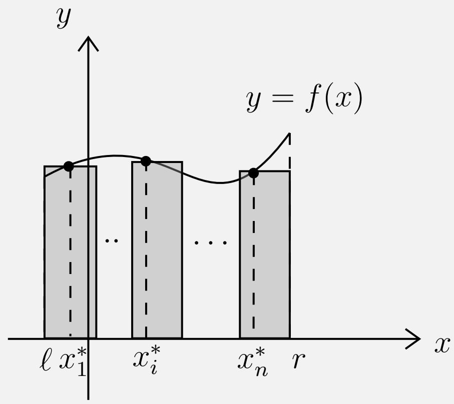
Definition 1. Let $f(x)$ to be defined on $[\ell, r]$. The *integral* is obtained in the following 4 steps.

1° Partition the interval $[\ell, r]$ into n subintervals $[x_{i-1}, x_i]$ with equal width $\Delta x = x_i - x_{i-1} = \dfrac{r-\ell}{n}$

2° Pick up all the sample (representative) points $x_i^* \in [x_{i-1}, x_i]$

3° Add up all the small rectangles $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x$ which is called *Riemann sum*

4° Take the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$



Then we say

- if the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and equals for all the choices of sample points, the $f(x)$ is integrable on $[\ell, r]$ and we use the notation $\displaystyle\int_{\ell}^{r} f(x)\,dx$.
- if the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ does no exist and does not equal for all the choices of sample points, the $f(x)$ is not integrable on $[\ell, r]$.

In practice, the difficulty of applying the definition is to show the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and equals for all the choices of sample points. One possible approach is to find the upper bound and lower bound of $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x$ and to apply the squeeze theorem. Let's look at an example.

Example 1. Prove $f(t) = t^2$ is integrable over $[0, 1]$.
Proof. First divide the domain into n subintervals $[t_{i-1}, t_i]$ with $t_i = \dfrac{i-1}{n}$. Then for every

sample point $t_{i-1} < t_i^* < t_i$, we have the bound of $f(t_i^*)$ that $\left(\frac{i-1}{n}\right)^2 < f(t_i^*) < \left(\frac{i}{n}\right)^2$. Then we add them up

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2 \leq \sum_{i=1}^{n} f(t_i^*)\Delta t \leq \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2$$

which could be simplified as

$$\underbrace{\frac{1}{n^3}\frac{(n-1)n(2n-1)}{6}}_{\to\frac{1}{3}\text{ when } n\to\infty} \leq \sum_{i=1}^{n} f(t_i^*)\Delta t \leq \underbrace{\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}}_{\to\frac{1}{3}\text{ when } n\to\infty}$$

Then we take the limit; by squeeze theorem, we finally have $\lim_{n\to\infty}\sum_{i=1}^{n} f(t_i^*)\Delta t = \frac{1}{3}$. This means the limit $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and equals for all the choices of sample points. Therefore, $f(t) = t^2$ is integrable over $[0,1]$.

We could generalize conclusion in the example to be a theorem.

Theorem 1. All continuous functions are integrable.

Then we are attracted to ask whether the discontinuous function integrable. Let's look at some examples.

Example 2. Let $f(t) = \begin{cases} 1 & t \neq 0 \\ 0 & t = 0 \end{cases}$. Prove $f(t)$ is integrable over $[-a,a]$ for $a > 0$.

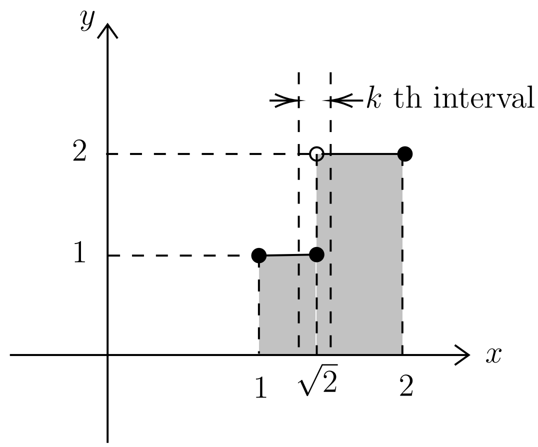
Proof. Again, we divide the interval into n subintervals. For the interval containing $t = 0$, $f(t_i^*)$ could be 0 or 1; otherwise $f(t_i^*) = 1$. Then we know

$$\frac{2a(n-1)}{n} = (n-1)\times\frac{2a}{n} + 0\times\frac{2a}{n} \leq \sum_{i=1}^{n} f(t_i^*)\Delta t \leq n\times\frac{2a}{n} = 2a$$

Then we take the limit, $\lim_{n\to\infty}\sum_{i=1}^{n} f(t_i^*)\Delta t = 2a$, which are equal for all choices of sample points.

Example 3. Let $f(t) = \begin{cases} 1 & 1 \leq t \leq \sqrt{2} \\ 2 & \sqrt{2} < t \leq 2 \end{cases}$. Prove $f(t)$ is integrable on $[1,2]$.

Proof. First divide the domain into n subintervals $[t_{i-1}, t_i]$ with $t_i = \frac{i-1}{n} + 1$. Then consider the k-th subinterval which contains $\sqrt{2}$.



We know $1+\dfrac{k-1}{n}\leq\sqrt{2}\leq 1+\dfrac{k}{n}$ which can be rearranged as

$$\sqrt{2}-1\leq\frac{k}{n}\leq\sqrt{2}-1+\frac{1}{n}$$

By squeeze theorem, we know $\lim\limits_{n\to\infty}\dfrac{k}{n}=\sqrt{2}-1$ and then $\lim\limits_{n\to\infty}\dfrac{k-1}{n}=\sqrt{2}-1$. Then we can consider the Riemann sum:

$$2-\frac{k}{n}=\frac{k}{n}+\frac{n-k}{n}\times 2\leq\sum_{i=1}^{n}f(t_i^*)\Delta t\leq\frac{k-1}{n}+\frac{n-k+1}{n}\times 2=2-\frac{k-1}{n}$$

Take the limit, by squeeze theorem, we have $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}f(t_i^*)\Delta t=\lim\limits_{n\to\infty}\left(2-\dfrac{k-1}{n}\right)=\lim\limits_{n\to\infty}\left(2-\dfrac{k-1}{n}\right)=3-\sqrt{2}$ for all choices of sample points. As a result, $\displaystyle\int_1^2 f(t)\,dt=3-\sqrt{2}$ which exists.

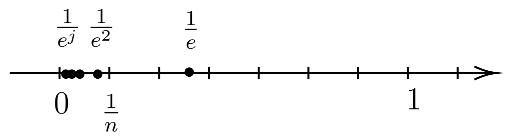
For the examples above, it seems that the jump or hole does not matter with integrablity. But when the discontinuous points become infinite, this claim may not always be true any more. Let's look at the following two examples.

Example 4. Let $f(t)=\begin{cases}1 & t\in\mathbb{Q}\\ 0 & t\in\mathbb{R}/\mathbb{Q}\end{cases}$. Prove $f(t)$ is not integrable over $[0,1]$.

Proof. Divide the $[\ell,r]$ into n subintervals $[t_{i-1},t_i]$ with $t_i=\dfrac{i}{n}$. Since $\dfrac{i-1}{n}<\dfrac{i+\frac{\sqrt{2}}{2}-1}{n}<\dfrac{i}{n}$ and $\dfrac{i-1}{n}<\dfrac{i+\frac{1}{2}-1}{n}<\dfrac{i}{n}$, for every subinterval we can find sample points $f(t_i^*)=1$ or $f(t_i^*)=0$. Then the limit of Riemann sum can be either $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}f(t_i^*)\Delta t=1$ with picking all the sample points to be 1 or $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}f(t_i^*)\Delta t=0$ with picking all the sample points to be 0. They are not equal for the different choices of sample points and therefore $f(t)$ is not integrable over $[0,1]$.

Example 5. Let $f(t) = \begin{cases} 1 & t = \frac{1}{e}, \frac{1}{e^2}, \frac{1}{e^3}, \cdots \\ 0 & \text{otherwise} \end{cases}$. Prove $f(t)$ is integrable over $[0,1]$.

Proof. First divide the interval $[0,1]$ into n subintervals. Then we want to find out how $\left\{\frac{1}{e^i}\right\}$ are distributed on $[0,1]$.



We could find there would be infinite $\frac{1}{e^j}$ located between $\left[0, \frac{1}{n}\right]$. Let k to the first term less than $\frac{1}{n}$. Then $\frac{1}{e^k} < \frac{1}{n} < \frac{1}{e^{k-1}}$ which implies

$$\log(n) < k < \log(n) + 1$$

Then consider the Riemann sum $\sum_{i=1}^{n} f(t_i^*)\Delta t$. The lower bound would be all the sample points $f(t_i^*) = 0$ and the upper bound would be all the possible sample points $f(t_i^*) = 1$ and other sample points $f(t_i^*) = 0$. Then we have

$$0 \leq \sum_{i=1}^{n} f(t_i^*)\Delta t \leq \frac{1}{n} + \frac{k-1}{n} < \frac{\log(n)+1}{n}$$

By L'Hospital rule, we have $\lim_{n\to\infty} \frac{\log(n)+1}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}+0}{1} = 0$. Therefore, by squeeze theorem, we know $\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\Delta t = 0$ for all choices of sample points. So $f(t)$ is integrable over $[0,1]$.

Then we should put our focus on the discontinuous points with asymptotes.

Example 6. Let $f(t) = \begin{cases} \frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$. Prove $f(t)$ is not integrable on $[0,1]$.

Proof. Divide $[0,1]$ into n subintervals $\left[\frac{i-1}{n}, \frac{i}{n}\right]$. Then for every subinterval we pick up the sample point with the smallest value $f(t_i^*)$. Then we know

$$\sum_{i=1}^{n} f(t_i^*) \geq 0 + \frac{1}{n}\sum_{i=1}^{n} \frac{n}{i} = \underbrace{\sum_{i=2}^{n} \frac{1}{i}}_{\text{diverges}}$$

Therefore the limit of the Riemann sum diverges and $f(t)$ is not integrable on $[0, 1]$.

This is also called improper integral and we are going to further discuss it.

2. **Properties**

   As we can see from the previous section, integral is not easy to compute based on definition. We want to find some short cut to simplify the integral. One way to do so is to decompose the integral.

   (a) **Zero Interval:** $\displaystyle\int_\ell^\ell f(x)dx = 0$

   Proof. Since $\Delta x = \dfrac{\ell - \ell}{n} = 0$, we have $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = 0$ and there is only one choice of $x_i^* = \ell$. So $\displaystyle\int_\ell^\ell f(x)dx = 0$.

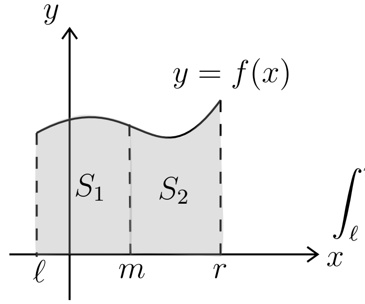
   (b) **Inverse Interval:** If $\displaystyle\int_\ell^r f(x)\,dx$ exists, then $\displaystyle\int_\ell^r f(x)\,dx = -\int_r^\ell f(x)\,dx$.

   Proof. That $\displaystyle\int_\ell^r f(x)\,dx$ exists implies $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ exists, where $\Delta x = \dfrac{r-\ell}{n}$. For $\int_r^\ell f(x)\,dx$, $\Delta x' = \frac{\ell - r}{n} = -\Delta x$. So $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x' = -\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ with corresponding choices of sample points and they are all equal. Therefore, $\displaystyle\int_\ell^r f(x)\,dx = -\int_r^\ell f(x)\,dx$.

   (c) **Interval Decomposition:** If $\displaystyle\int_\ell^r f(x)\,dx$, $\displaystyle\int_\ell^m f(x)\,dx$ and $\displaystyle\int_m^r f(x)\,dx$ exist, then $\displaystyle\int_\ell^r f(x)\,dx = \int_\ell^m f(x)\,dx + \int_m^r f(x)\,dx$.

   Proof. Without loss of generality, by (b), let $\ell < r$. We treat the integral $\displaystyle\int_\ell^r f(x)\,dx$ as area. There are three cases: $\ell < m < r$, $m < \ell < r$ and $\ell < r < m$.

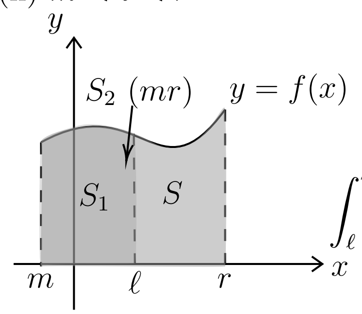
   (i) $\ell < m < r$



$$S = S_1 + S_2$$

$$\int_\ell^r f(x)\ dx = \int_\ell^m f(x)\ dx + \int_m^r f(x)dx$$

(ii) $m < \ell < r$

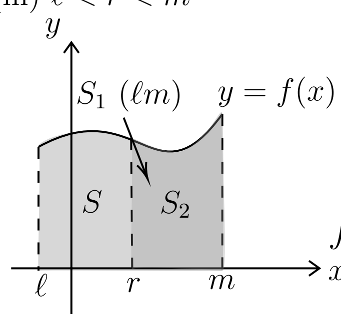


$$S = S_2 - S_1$$

$$\Updownarrow$$

$$\int_\ell^r f(x)\ dx \ = \int_m^r f(x)\ dx - \int_m^\ell f(x)dx$$
$$= \int_m^r f(x)dx + \int_\ell^m f(x)dx$$

(iii) $\ell < r < m$



$$S = S_1 - S_2$$

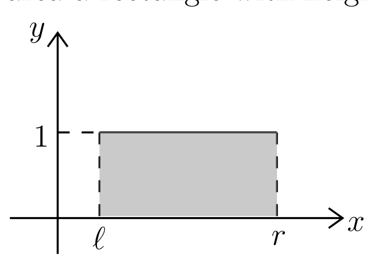
$$\Updownarrow$$

$$\int_\ell^r f(x)\ dx \ = \int_\ell^m f(x)\ dx - \int_r^m f(x)dx$$
$$= \int_\ell^m f(x)\ dx + \int_m^r f(x)\ dx$$

In summary, $\displaystyle\int_\ell^r f(x)\,dx = \int_\ell^m f(x)\,dx + \int_m^r f(x)\,dx$.

(d) **Integral over One:** $\displaystyle\int_\ell^r 1\,dx = r - \ell$

Proof. By definition, $\displaystyle\int_\ell^r 1\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^n \Delta x = \lim_{n\to\infty} n \times \frac{r-\ell}{n} =$ $r - \ell$. In the graph, it is the area a rectangle with height 1 and width $r - \ell$.

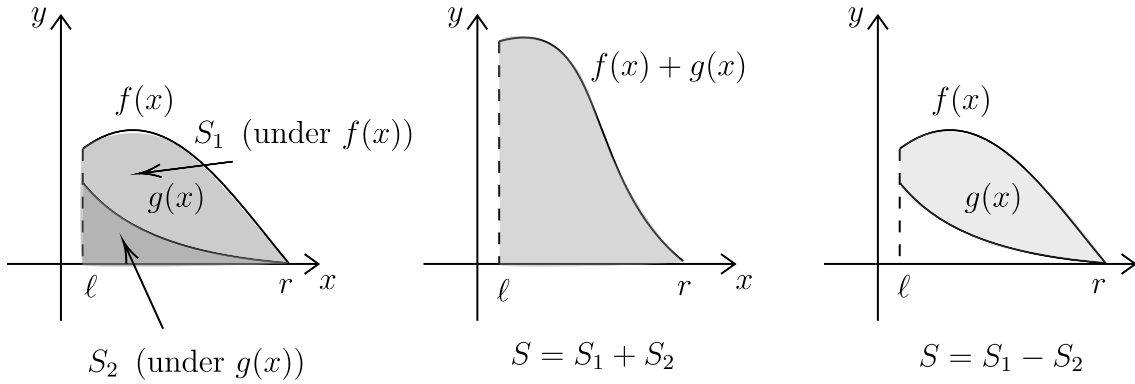


(e) **Integral with Constant Coefficient:** If $\displaystyle\int_r^\ell f(x)\,dx$ exists, $\displaystyle\int_\ell^r cf(x)\,dx = c\int_\ell^r f(x)\,dx$.

Proof. By definition, $\displaystyle\int_\ell^r cf(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n cf(x_i^*)\Delta x = c\lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x = c\int_\ell^r f(x)\,dx$ where $\Delta x = \dfrac{r-\ell}{n}$ with same choices of sample points.

Corollary: $\displaystyle\int_\ell^r c\,dx = c(r - \ell)$

(f) **Integral of the Sum of Functions:** If $f(x)$ and $g(x)$ are integrable on $[\ell, r]$, $\displaystyle\int_\ell^r [f(x) \pm g(x)]\, dx = \int_\ell^r f(x)\, dx \pm \int_\ell^r g(x)\, dx$.



Proof. By definition of integral, $\displaystyle\int_\ell^r [f(x) \pm g(x)]\, dx = \lim_{n\to\infty} \sum_{i=1}^n [f(x_i^*) + g(x_i^*)]\, \Delta x =$ $\displaystyle\lim_{n\to\infty} \left(\sum_{i=1}^n f(x_i^*)\Delta x + \sum_{i=1}^n g(x_i^*)\Delta x \right) = \underbrace{\lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x + \lim_{n\to\infty} \sum_{i=1}^n g(x_i^*)\Delta x}_{\text{both limits exist by definition}} = \int_\ell^r f(x)\, dx \pm$ $\displaystyle\int_\ell^r g(x)\, dx$ where $\Delta x = \dfrac{r - \ell}{n}$ with same choices of sample points.

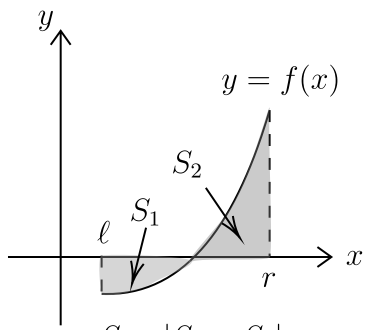
(g) **Sign Preserving:** If $f(x)$ is integrable on $[\ell, r]$ and $f(x) \geq 0$, then $\displaystyle\int_\ell^r f(x)\, dx \geq 0$.

Proof. By definition, $\displaystyle\int_\ell^r f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^n \underbrace{f(x_i^*)}_{f(x)\geq 0} \overbrace{\Delta x}^{\frac{\ell - r}{n} > 0} \geq 0$.

Corollary: If $f(x)$ and $g(x)$ are integrable on $[\ell, r]$ and $f(x) \geq g(x)$, then $\displaystyle\int_\ell^r f(x)\, dx \geq \int_\ell^r g(x)\, dx$.

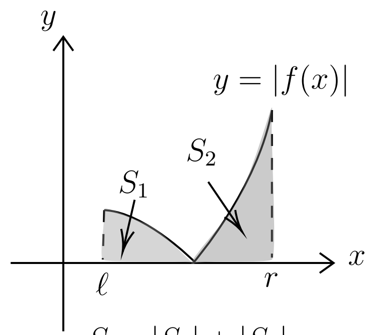
Proof. Since $f(x) - g(x) \geq 0$, by definition, we have $\displaystyle\int_\ell^r f(x)\, dx - \int_\ell^r g(x)\, dx =$ $\displaystyle\int_\ell^r \underbrace{[f(x) - g(x)]}_{\geq 0}\, dx \geq 0$. Therefore, $\displaystyle\int_\ell^r f(x)\, dx \geq \int_\ell^r g(x)\, dx$.

Corollary: If $f(x)$ and $|f(x)|$ are integrable on $[\ell, r]$, then $\displaystyle\left| \int_\ell^r f(x)\, dx \right| \leq \int_\ell^r |f(x)|\, dx$.



$S = |S_2 - S_1|$

Net Area

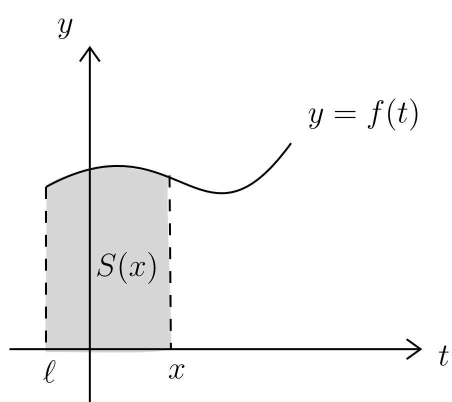


$S = |S_1| + |S_2|$

Total Area

Proof. Since $-|f(x)| \le f(x) \le |f(x)|$, we have $-\int_\ell^r |f(x)|\,dx \le \int_\ell^r f(x)\,dx \le \int_\ell^r |f(x)|\,dx$. This is $\left|\int_\ell^r f(x)\,dx\right| \le \int_\ell^r |f(x)|\,dx$.

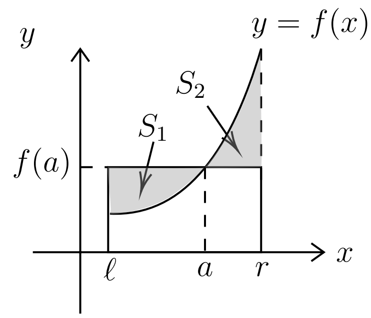
3. **Fundamental Theorem of Calculus**
   With the properties as decomposition tools above, we still can not avoid the definition to calculate the integral so we want to find a new tool. One guess is to make connection to derivative to find out how $f(x)$ is integrated along x. Let $F(x) = \int_\ell^x f(t)\,dt$. Then what is $\dfrac{d}{dx}F(x)$?



$$S(x) = F(x) = \int_\ell^x f(t)\ dt$$

Let's put it into a real world scenario. Let $f(t) = v(t)$, the velocity of the object at t. Let $s(t) = F(t)$ (copying the corresponding rule F), the distance the object has travelled until t. From practice, $s'(t) = v(t)$ which is $F'(x) = f(x)$. This is reasonable. Consider $f(x)$ as the rate of area $F(x)$ increasing, $F'(x) = f(x)$. To prove this, let's first prove a premise of it.

Theorem 2. (Mean Value Theorem of Integrals) Let $f(x)$ be continuous on $[\ell, r]$. There exists a number $a \in [\ell, r]$ such that $f(a)(r-\ell) = \int_\ell^r f(x)\,dx$.



This theorem could be interpreted in two ways. One direct way is that we can find a number $a \in [\ell, r]$ such that the integral equals to the area of a rectangle with height $f(a)$ and width $r - \ell$. Then we could say $f(a)$ is the average (mean) height of the function $f(x)$. Another way of interpretation is that there exists a number $a \in [\ell, r]$ such that $S_1 = S_2$ labeled in the graph.

Proof. Since $f(x)$ is continuous on $[\ell, r]$, by extreme value theorem, $f(x)$ has a global maximum $f(x_1) = U$ and a global minimum $f(x_2) = L$. So that $L \leq f(x) \leq U$ and $x_1,\, x_2 \in [\ell, r]$. This implies

$$L(r-\ell) \leq \int_\ell^r f(x)\,dx \leq U(r-\ell)$$

then

$$L \leq \frac{1}{r-\ell}\int_\ell^r f(x)\,dx \leq U$$

By intermediate value theorem, there exists a number a between x_1 and x_2 (therefore $a \in [\ell, r]$) such that $f(a) = \dfrac{1}{r-\ell}\displaystyle\int_\ell^r f(x)\,dx$.

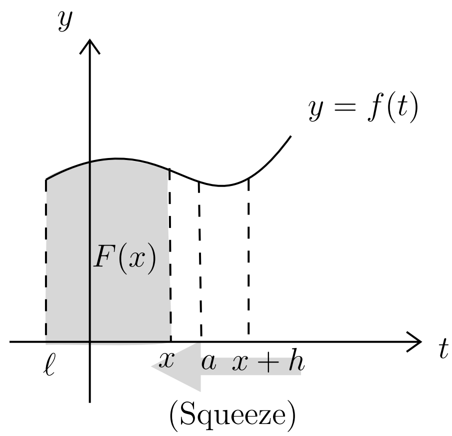
Then let's explore the relationship between $f(x)$ and $F'(x)$ - the fundamental theorem of calculus.

Theorem 3. (Fundamental Theorem of Calculus) Let $f(x)$ to be continuous on an interval I containing the point t and let $F(x) = \int_\ell^x f(t)\,dt$, then:

(a) $F'(x) = f(x)$.

(b) Let $G(x)$ to be the antiderivative (such that $G'(x) = f(x)$) of $f(x)$, then $\int_\ell^r f(x)\,dx = G(r) - G(\ell)$ for all $r \in I$.

Proof. (a) Consider the difference between $F(x)$ and $F(x+h)$,



by definition, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{\int_\ell^{x+h} f(t)\,dt - \int_\ell^{x} f(t)\,dt}{h} \\ &= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} \\ &= \lim_{h\to 0} \frac{f(a)\cancel{h}}{\cancel{h}}\ (a \in [x, x+h]) && \text{by MVT of Integrals} \\ &= f(x) && \text{by squeeze theorem} \end{aligned}$$

(b) By (a), we know the antiderivative exists. Since $F'(x) = G'(x) = f(x)$, we let $G(x) = F(x) + C$. Then we have $\begin{cases} G(r) &= F(r) + C \\ G(\ell) &= F(\ell) + C = C \end{cases}$. Therefore $G(r) - G(\ell) = F(r) + C - C = \displaystyle\int_\ell^r f(x)\,dx$.

Corollary: (Extensions of the Mean Value Theorem of Integrals) Let $f(x)$ continuous on $[\ell, r]$. There exists $a \in (\ell, r)$ such that $\displaystyle\int_\ell^r f(x)\,dx = f(a)(\ell - r)$.

Proof. Let $F(x) = \displaystyle\int_\ell^x f(t)\,dt$. By fundamental theorem of calculus, $F'(x) = f(x)$. And we have $\displaystyle\int_\ell^r f(x)\,dx = F(r) - F(\ell) = \underbrace{F'(a)(\ell - r)}_{\ell < a < r,\ \text{by MVT}} = f(a)(\ell - r)$.

Note: The continuity of $f(x)$ is the key of FTC to be true. Look at example 5, $F(x) = \displaystyle\int_0^x f(t)\,dt = 0$. However $\dfrac{d}{dx}F(x) = 0 \neq f(x)$.

Let's check two examples.

Example 7. Evaluate $\displaystyle\int_0^1 x\,dx$. By fundamental theorem of calculus, we have $\displaystyle\int_0^1 x\,dx =$

$\frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$.

Example 8. Evaluate $\int_0^{\pi} \sin x\, dx$. By fundamental theorem of calculus, we have $\int_0^{\pi} \sin x\, dx$ $= -\cos x|_0^{\pi} = 2$

4. **Improper Integral**

   Recall the definition of integral: the integrand is defined to be defined on a finite closed interval $[\ell, r]$. Then what about the integral with infinite interval or having asymptote in the interval? This kind of integral are called improper integral.

   > **Definition 2.** An *improper integral* is a definite integral that has either or both limits infinite or an integrand that approaches infinity at one or more points in the range of integration.

   It is obvious that the improper integral can not be computed by Riemann sum. Then the approach is based on the combination of fundamental theorem of calculus and limit.

   For example, let $f(x)$ defined on (ℓ, ∞) and $\lim_{x\to\ell^+} f(x) = +\infty$. Then with $\ell < a < b < \infty$, we have $\int_\ell^\infty f(x)\, dx = \lim_{a\to\ell^+}\lim_{b\to\infty}\int_a^b f(x)\, dx = \lim_{a\to\ell^+}\lim_{b\to\infty}[F(b) - F(a)] = \lim_{b\to\infty} F(b) - \lim_{a\to\ell^+} F(a)$ where $F(x)$ is the antiderivative of $f(x)$. If one of the limits does not exist, the integral does not exist.

Problem Solving I

Prove integrablity by definition; Calculate limit by definition of integral; Calculation related to $F(x) = \int_\ell^x f(t)\,dt$; improper integral

Problem 1: Prove integrablity by definition

1. Prove $\displaystyle\int_1^2 t\,dt = \frac{3}{2}$.
2. Based on the fact $f(t) = 1 - 2t$ is continuous, compute $\displaystyle\int_0^2 (1-2t)\,dt$.
3. Suppose $f(t)$ is integrable and negative on the interval $[0,1]$. Let $g(t) = \begin{cases} f(t) & 0 \le t < 1 \\ 2 & t = 1 \end{cases}$.

   Prove $g(t)$ is integrable on $[0,1]$.
4. Let $f(t) = \begin{cases} A & t = k \\ B & \text{otherwise} \end{cases}$, where A, B and k are constants with $A < B$. Prove that $f(t)$ is integrable on any finite interval $[\ell, r]$.
5. Let

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{j}{2^k} \text{ for integer } j \text{ and } k\text{, with } k \text{ positive and } 0 \le j \le 2^k \\ -1 & \text{otherwise} \end{cases}$$

   Prove $f(x)$ is not integrable on $[0,1]$.

Problem 2: Calculate limit by definition of integral

Let $f(x)$ integrable on $[0,1]$. Then $\displaystyle\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1} f\left(\frac{i-1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1} f\left(\frac{i}{n}\right)$. Based on that, compute the following limit.

1. $\displaystyle\lim_{n\to\infty} \frac{i^4}{n^5}$
2. $\displaystyle\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right)$
3. $\displaystyle\lim_{n\to\infty} \frac{1^3 + 2^3 + 3^3 + \cdots + n^3}{n^4}$
4. $\displaystyle\lim_{n\to\infty} \sum_{k=1}^n \frac{6(k-1)^2}{n^3} \sqrt{1 + 2\frac{(k-1)^3}{n^3}}$

Problem 3: Calculation related to $F(x) = \int_\ell^x f(t)\,dt$

1. Let $f(x)$ continuous and $\varphi_1(x)$, $\varphi_2(x)$ are differentiable. Compute $\dfrac{d}{dx}\displaystyle\int_{\varphi_2(x)}^{\varphi_1(x)} f(t)\,dt$.
2. Compute $\dfrac{d}{dx}\displaystyle\int_1^x e^{-t^2}\,dt$.
3. Compute $\dfrac{d}{dx}\left(\sin x \displaystyle\int_{x^2}^{x^3} e^{-t^2}\right)$.
4. Compute $\dfrac{d}{dx}\left(e^x \displaystyle\int_{x^2}^{x^5} \cos(t^2)\,dt\right)$.
5. Solve $3 + \displaystyle\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ to find a and $f(x)$.
6. Let $F(x) = \displaystyle\int_1^x f(t)\,dt$, where $f(t) = \displaystyle\int_1^{t^2} \frac{\sqrt{1+u^2}}{u}\,du$. Find $F''(2)$.
7. If $x\sin(\pi x) = \displaystyle\int_0^{x^2} f(t)\,dt$ and $f(t)$ is continuous, find $f(4)$.
8. Solve $f(x) = 1 + 4\displaystyle\int_3^x f(t)\,dt$ to find $f(x)$.
9. Find the global extreme of $F(x) = \int_0^{2x-x^2} \cos\left(\frac{1}{1+t^2}\right)$.
10. Evaluate $\displaystyle\lim_{x\to 0} \frac{\int_0^x (1-\tan(2t))^{\frac{1}{t}}\,dt}{x}$

Problem 4: Improper Integral

1. Write the definition of $\displaystyle\int_\ell^\infty f(t)\,dt$.
2. Write the definition of $\displaystyle\int_{0^+}^1 \frac{1}{\sqrt{t}}\,dt$.
3. Prove that $f(t) = \dfrac{1}{t^2}$ is integrable over $[1, +\infty)$.
4. Evaluate $\displaystyle\int_1^\infty e^{-t}\,dt$ and estimate $\displaystyle\int_1^\infty e^{-t^2}\,dt$.
5. Evaluate $\displaystyle\int_{-\infty}^\infty 8e^{-|x|}\,dx$.
6. Prove that $\displaystyle\int_0^r t^{-p}$ where $r > 0$ converges if and only if $p < 1$.

7. Prove that $\displaystyle\int_\ell^\infty t^{-p}$ where $\ell > 0$ converges if and only if $p > 1$.

Integral Techniques: Substitution, Parts and Partial Fraction

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Based on the fundamental theorem of calculus, to find the integral we only have to find the antiderivative. With the basic derivative formula we could find the antiderivative of the basic functions. However, when we meet the combination of basic functions, we need some new techniques. A good clue to find them is to use the derivative rules.

1. **Integration by Substitution**
   Consider chain rule,
   $$F(\varphi(x)) = f[\varphi(x)]\varphi'(x)$$
   where $F'(x) = f(x)$. If we note the antiderivative of $f(x)$ as $\displaystyle\int f(x)\,dx$, we have
   $$\int f[\varphi(x)]\varphi'(x)\,dx = F(\varphi(x)) + C$$
   Then by fundamental theorem of calculus, we have
   $$\int_a^b f[\varphi(x)]\varphi'(x)\,dx \;\overset{t=\varphi(x)}{\underset{dt=\varphi'(x)dx}{=\!=\!=}}\; \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt = F(t)|_{\varphi(a)}^{\varphi(b)} = F(\varphi(t))|_a^b \tag{S1}$$
   This could also be done in another direction if it gives a simpler form to find antiderivatives. We have
   $$\int f(x)\,dx \overset{x=\varphi(t)}{=\!=\!=} \int \underbrace{f(t)\varphi'(t)}_{g(t)}\,dt = G[\varphi^{-1}(t)] + C$$
   where $G'(t) = g(t)$. Then for the integral, we have
   $$\int_a^b f(x)\,dx \overset{x=\varphi(t)}{=\!=\!=} \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} g(t)\,dt = G(t)|_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} \tag{S2}$$
   Then let's look at some examples where we apply (S1).

   Example 9. Consider the following integrals:
   - $\displaystyle\int_0^1 \cos(2t+1)\,dt = \frac{1}{2}\int_0^1 \cos(2t+1)\,d(2t+1) \overset{u=2t+1}{=\!=\!=} \frac{1}{2}\int_1^5 \cos(x)\,dx = \frac{1}{2}\sin(x)\Big|_1^5 = \frac{\sin(5)-\sin(1)}{2}$
   - $\displaystyle\int_0^2 \frac{t}{t^2+3}\,dt = \int_0^2 \frac{1}{2(t^2+3)}\,d(t^2+3) \overset{u=t^2+3}{=\!=\!=} \int_2^7 \frac{1}{2u}\,du = \frac{1}{2}\log(u)\Big|_3^7 = \frac{1}{2}\log\left(\frac{7}{3}\right)$
   - $\displaystyle\int_1^2 \frac{1}{\sqrt{x}(1+x)}\,dx = 2\int_1^2 \frac{1}{1+(\sqrt{x})^2}d(\sqrt{x}) = 2\arctan\sqrt{x}\big|_1^2 = 2\arctan(\sqrt{2}) - 2\arctan(1)$
   - $\displaystyle\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\,dx = \int_0^{\frac{\pi}{2}} \sin(x)\,d(\sin(x)) = \frac{\sin^2(x)}{2}\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}$

Actually, the last integral in the example is a very special example related to to sin and cos functions. We could explore more examples and summarize a general approach to this kind of integral.

Example 10. Consider the following integrals:

- $\displaystyle\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\,dx = \int_0^{\frac{\pi}{2}} \sin(x)\,d(\sin(x)) = \frac{\sin^2(x)}{2}\bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}$
- $\displaystyle\int_0^{\frac{\pi}{2}} \sin(t)\cos^4(t)\,dt = -\int_0^{\frac{\pi}{2}} \cos^4(t)d(\cos(t))\,dt = -\frac{\cos^5(t)}{4}\bigg|_0^{\frac{\pi}{2}} = \frac{1}{4}$
- $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2(t)\cos^2(t)\,dt = \int_0^{\frac{\pi}{2}} \frac{1-\cos(2t)}{2} \times \frac{1+\cos(2t)}{2}\,dt = \frac{1}{4}\int_0^{\frac{\pi}{2}} (1-\cos^2(2t))\,dt = \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2(2t)\,dt = \frac{1}{8}\int_0^{\frac{\pi}{2}} (1-\cos(4t))\,dt = \left(\frac{t}{8} - \frac{1}{32}\sin(4t)\right)\bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{16}$
- $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2(t)\,dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2t)}{2}\,dt = \frac{1}{2}t + \frac{1}{4}\cos(2t)\bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

In summary, this type of integrand is $f(t) = \sin^m(t)\cos^n(t)$ where $m, n \in \mathbb{Z}$. If m is odd, then let $u = \cos(t)$, we could have $\displaystyle\int \sin^m(t)\cos^n(t)\,dt = \int -(1-u^2)^{\frac{m-1}{2}}u^n\,du$. If n is odd, then let $u = \sin(t)$, we could have $\displaystyle\int \sin^m(t)\cos^n(t)\,dt = \int u^m(1-u)^{\frac{n-1}{2}}\,du$. If m, n are both odd, then pull out the small one as $u'(t)$. If m, n are both even, then use $cos^2(t) = \dfrac{1-\cos(2t)}{2}$ and $sin^2(t) = \dfrac{1+\cos(2t)}{2}$ to reduce the order repeatedly.

Then let's look at some examples applying (S2).

Example 11. Consider the following integral:

- $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx \overset{x=\sin t}{=\!=\!=} \int_0^{\arcsin 1} \cos t\cos t\,dt = \int_0^{\arcsin 1} \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\,t + \frac{1}{2}\sin 2t\bigg|_0^{\arcsin 1} = \frac{1}{2}\,(t + \sin t\cos t)\bigg|_0^{\arcsin 1} = \frac{\arcsin 1}{2}$
- $\displaystyle\int_2^3 \frac{1}{\sqrt{x^2-1}}\,dx \overset{x=\sec t}{=\!=\!=} \int_{\operatorname{arcsec} 2}^{\operatorname{arcsec} 3} \frac{\sec t\tan t}{\tan t}\,dt = \int_{\operatorname{arcsec} 2}^{\operatorname{arcsec} 3} \sec t\,dt = \int_{\operatorname{arcsec} 2}^{\operatorname{arcsec} 3} \frac{1}{1-\sin^2 t}\,d(\sin t) = \int_{\operatorname{arcsec} 2}^{\operatorname{arcsec} 3} \frac{1}{2}\left(\frac{1}{1-\sin t} + \frac{1}{1+\sin t}\right)d(\sin t) = \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right|\Bigg|_{\operatorname{arcsec} 2}^{\operatorname{arcsec} 3} = \frac{1}{2}\ln\left|\frac{1+2\sqrt{2}}{1-2\sqrt{2}}\cdot\frac{1-\sqrt{3}}{1+\sqrt{3}}\right| = \frac{1}{2}\ln\left|\frac{1-2\sqrt{6}+2\sqrt{2}-\sqrt{3}}{1-2\sqrt{6}-2\sqrt{2}+\sqrt{3}}\right|$
- $\displaystyle\int_0^1 \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx \overset{x=\tan t}{=\!=\!=} \int_0^{\arctan 1} \frac{\sec^2 t}{\sec^3 t}\,dt = \int_0^{\arctan 1} \cos t\,dt = \sin t|_0^{\arctan 1} = \frac{\sqrt{2}}{2}$

In summary, here we use two powerful formulas: $\sin^2 t + \cos^2 t = 1$ and $\tan^2 t + 1 = \sec^2 t$. By them, we have three substitutions:

- $\sqrt{a^2 - x^2} \xlongequal{x=a\sin\theta} a\cos\theta$ ($\sqrt{a^2 - x^2} \xlongequal{x=a\cos\theta} a\sin\theta$)
- $\sqrt{x^2 - a^2} \xlongequal{x=\sec\theta} a\tan\theta$
- $\sqrt{x^2 + a^2} \xlongequal{x=\tan\theta} a\sec\theta$

Besides those specific integral, we could use substitution to explore the properties of some general integrals. For example let's consider the integral with symmetric interval. Let $f(x)$ integrable on $[-a, a]$. We have

$$\int_{-a}^{0} f(x)\,dx \xlongequal{x=-t} \int_{0}^{a} f(-t)\,dt \xlongequal{\text{dummy variable:}t=x} \int_{0}^{a} f(x)\,dx$$

Then we would have

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} [f(-x) + f(x)]\,dx\end{aligned}$$

If $f(x) = -f(-x)$, we have $\int_{-a}^{a} f(x)\,dx = 0$. If $f(x) = f(-x)$, we have $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Besides, we could also explore the integral properties of periodic functions. Let $f(x)$ integrable and $f(x+T) = f(x)$. We have

$$\int_{T}^{a+T} f(x)\,dx \xlongequal{x=T+t} \int_{0}^{a} f(T+t)\,dt = \int_{0}^{a} f(t)\,dt \xlongequal{\text{dummy variable:}t=x} \int_{0}^{a} f(x)\,dx$$

Then

$$\begin{aligned}\int_{a}^{a+T} f(x)\,dx &= \int_{a}^{0} f(x)\,dx + \int_{0}^{T} f(x)\,dx + \int_{a}^{a+T} f(x)\,dx \\ &= \cancel{\int_{a}^{0} f(x)\,dx} + \int_{0}^{T} f(x)\,dx + \cancel{\int_{0}^{a} f(x)\,dx} \\ &= \int_{0}^{T} f(x)\,dx\end{aligned}$$

2. **Integral by Parts**
Recall the product rule,

$$(uv)' = uv' + u'v$$

Rearrage it, we have

$$uv' = (uv)' - u'v$$

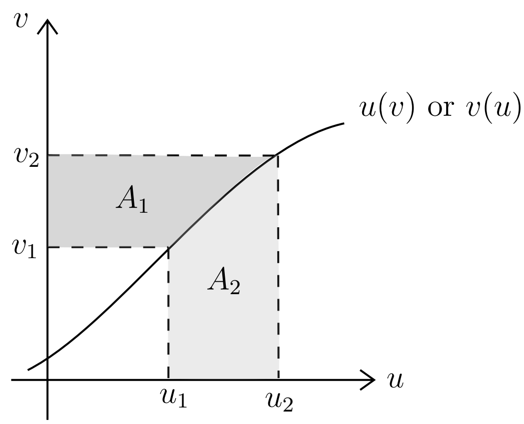
Find the antiderivative on both sides, we have

$$\int u dv = uv - \int v du + C$$

By fundamental theorem of calculus, we can find the integral

$$\int_a^b u dv = uv|_a^b - \int_a^b v du$$

We could visualize this technique. Consider the following diagram.



We know the area of $A_1 + A_2$ is the area of the big rectangle minus the area of the big rectangle. This is just

$$\underbrace{\int_{v_1}^{v_2} u(v)\, dv}_{A_1} + \underbrace{\int_{u_1}^{u_2} v(u)\, du}_{A_2} = u_2 v_2 - v_1 v_1$$

or, in terms of antiderivative,

$$\int u\, dv + \int v\, du = uv + C$$

By arrangement, we have

$$\int u\, dv = uv - \int v\, du + C$$

This is a powerful tool to evaluate complicated integral $\displaystyle\int_a^b u\, dv$ by simpler integral $\displaystyle\int_a^b v du$. Let's look at some examples.

Example 12. Consider the following integrals:

- $\displaystyle\int_1^2 \ln x\, dx = x \ln x|_1^2 - \int_1^2 x d(\ln x) = x \ln x - x|_1^2 = 2 \ln 2 - 1$

- $\int_1^2 xe^x\,dx = xe^x|_1^2 - \int_1^2 e^x dx = xe^x - e^x|_1^2 = e^2$
- $\int_0^\pi t\sin t\,dt = \int_0^\pi td(\cos t) = t\cos t|_0^\pi - \int_0^\pi \cos t\,dt = -2\pi - 2\sin t|_0^\pi = -2\pi$
- $I = \int_0^\pi e^x\sin x\,dx = \int_0^\pi \sin x\,d(e^x) = \sin xe^x|_0^\pi - \int_0^\pi \cos xe^x\,dx = -\int_0^\pi \cos x\,d(e^x) = -e^x\cos x|_0^\pi - \int_0^\pi e^x\cos x\,dx = e^\pi + 1 - I$. Then $I = \dfrac{e^\pi+1}{2}$.
- $\int_0^{\frac{\sqrt{2}}{2}} \arcsin x\,dx = x\arcsin x|_0^{\frac{\sqrt{2}}{2}} - \int_0^{\frac{\sqrt{2}}{2}} xd(\arcsin x) = \dfrac{\sqrt{2}\pi}{8} - \int_0^{\frac{\sqrt{2}}{2}} \dfrac{x}{\sqrt{1-x^2}}\,dx = \dfrac{\sqrt{2}\pi}{8} + \int_0^{\frac{\sqrt{2}}{2}} \dfrac{d(1-x^2)}{2\sqrt{1-x^2}} = \dfrac{\sqrt{2}\pi}{8} + \sqrt{1-x^2}|_0^{\frac{\sqrt{2}}{2}} = \dfrac{\sqrt{2}\pi}{8} + \dfrac{\sqrt{2}}{2} - 1$
- $I = \int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \int_0^{\frac{\pi}{4}} \sec\theta\,d(\tan\theta) = \sec\theta\tan\theta|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2\theta\sec\theta\,d\theta = \sec\theta\tan\theta|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (\sec^3\theta - \sec\theta)\,d\theta = \sqrt{2} - I + \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta = \sqrt{2} - I + \dfrac{1}{2}\ln\left|\dfrac{1+\sin\theta}{1-\sin\theta}\right|\Bigg|_0^{\frac{\pi}{4}} = \sqrt{2} + \ln(\sqrt{2}+2) - I$. Then $I = \dfrac{\sqrt{2}+\ln(\sqrt{2}+2)}{2}$.

In summary, integral by parts mainly deals with the multiplication of $\ln x$, $\sin x$, $\cos x$, $\tan x$, e^x, $\arcsin x$, $\arccos x$ and x^n.

3. **Integral by Partial Fraction**

Consider $f(x) = \dfrac{h(x)}{g(x)}$ where $h(x)$ and $g(x)$ are polynomials and the degree of $h(x)$ is lower than the degree of $g(x)$. We want to simplify or decompose $f(x)$ to make the integral of $f(x)$ easier. We can represent $g(x)$ as

$$g(x) = kL_1^{m_1}L_2^{m_2}\cdots L_a^{m_a}Q_1^{n_1}Q_2^{n_2}\cdots Q_b^{n_b}$$

where L_i are linear terms as $x-a$ and Q_j are irreducible quadratic factors as $x^2+bx+c\,(b^2-4c<0)$. Then we can decompose $f(x)$ as

$$f(x) = \sum_{k=1}^{a}\left(\frac{A_1}{L_k} + \frac{A_2}{L_k^2} + \cdots \frac{A_{m_i}}{L_k^{m_i}}\right) + \sum_{k=1}^{b}\left(\frac{B_1x+C_1}{Q_k} + \frac{B_2x+C_2}{Q_k^2} + \cdots \frac{B_{n_i}x+C_{n_i}}{Q_k^{n_i}}\right)$$

This is called Euclidean algorithm. Then we could integrate $f(x)$ by small pieces.

Let's look at an example.

Example 13. Evaluate $\displaystyle\int_0^1 \frac{2x^3+11x^2+28x+33}{x^2-x-6}\,dt.$
We could make a long division to make the order of numerator is lower than the denominator.

$$
\begin{array}{r}
2x \quad + 13 \\
x^2 - x - 6 \overline{\smash{\big)}\; 2x^3 + 11x^2 + 28x \quad + 33} \\
\underline{-\, 2x^3 \quad + 2x^2 + 12x} \qquad\; \\
13x^2 + 40x \quad + 33 \\
\underline{-\, 13x^2 + 13x \quad + 78} \\
53x + 111
\end{array}
$$

Then we know $f(x) = 2t + 13 + \dfrac{53t + 11}{t^2 - t - 6} = 2t + 13 + \dfrac{A}{t-3} + \dfrac{B}{t+2}$. Then $A(t+2) + B(t-3) = (A+B)t + 2A - 3B = 53t + 11$ which gives $\begin{cases} A + B = 53 \\ 2A - 3B = 111 \end{cases}$. Then we know $A = 54$ and $B = -1$. Therefore we know the integral is

$$
\begin{aligned}
\int_0^1 \frac{2x^3 + 11x^2 + 28x + 33}{x^2 - x - 6}\, dt &= \int_0^1 \left(2t + 13 + \frac{54}{t-3} - \frac{1}{t+2} \right) dt \\
&= \left(t^2 + 13t + 54 \ln|t-3| - \ln|t+3| \right)\Big|_0^1 \\
&= 14 + 55 \ln \frac{2}{3}
\end{aligned}
$$

Problem Solving II

Evaluate the integral; Explore the general properties of integral

Problem 5: Evaluate the integral

- **Integral by Substitution**

  1. Evaluate the following integral

     (a) $\displaystyle\int_1^2 \frac{1}{x}\cos\left(\frac{1}{x^2}\right)\,dx$

     (b) $\displaystyle\int_0^1 \frac{e^x}{1+e^{2x}}\,dx$

     (c) $\displaystyle\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{1}{\sqrt{x(1-x)}}\,dx$

     (d) $\displaystyle\int_0^4 \frac{1}{\sqrt{25-x^2}}\,dx$

     (e) $\displaystyle\int_0^1 \frac{1}{x^2+25}\,dx$

     (f) $\displaystyle\int_1^2 \frac{x^2}{(x+2)^3}\,dx$

     (g) $\displaystyle\int_0^1 \frac{x^2+1}{x^4+1}\,dx$

     (h) $\displaystyle\int_5^6 \frac{2t}{\sqrt{t-4}}$

  2. Evaluate the following integral

     (a) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$

     (b) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\cos^5 x\,dx$

     (c) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

     (d) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\cos^4 x\,dx$

     (e) $\displaystyle\int_0^{\frac{\pi}{4}} \tan x\,dx$

     (f) $\displaystyle\int_0^{\frac{\pi}{4}} \csc x\,dx$

     (g) $\displaystyle\int_0^{\frac{\pi}{4}} \sec x\,dx$

(h) $\int_0^{\frac{\pi}{4}} \tan^4 x \, dx$

(i) $\int_0^{\frac{\pi}{4}} \sec^5 x \tan x \, dx$

(j) $\int_0^{\frac{\pi}{4}} \frac{1}{1+\cos x} \, dx$

3. Evaluate the integral

(a) $\int_0^{a} b\sqrt{1-\frac{x^2}{a^2}} \, dx$ ($a, b > 0$ are real numbers)

(b) $\int_1^{2} \frac{1}{x^2\sqrt{x^2+1}} \, dx$

(c) $\int_0^{1} \frac{t^3}{(4x^2+9)^{\frac{3}{2}}} \, dx$

(d) $\int_1^{4} \sqrt{t^2-1} \, dt$

(e) $\int_1^{2} \sqrt{4-x^2} \, dx$

(f) $\int_0^{1} \frac{1}{(4x^2+9)^{\frac{3}{2}}} \, dx$

(g) $\int_0^{\frac{1}{3}} \frac{18}{1-2x^2} \, dx$

(h) $\int_0^{1} \frac{1}{\sqrt{4x^2+9}} \, dx$

(i) $\int_1^{2} \frac{1}{x^2+x} \, dx$

- **Integral by Parts**

4. Evaluate the following integrals: $x^n \ln^\alpha x$ where α would be the times of using by parts

(a) $\int_1^{e} \frac{\ln x}{\sqrt{x}} \, dx$

(b) $\int_1^{e} \ln x \, dx$

(c) $\int_1^{e} \ln^2 x \, dx$

(d) $\int_1^{e} x^2 \ln x \, dx$

5. Evaluate the following integrals: $x^\alpha e^x$ where α would be the times of using by parts

(a) $\int_0^1 xe^x\,dx$

(b) $\int_0^1 x^2e^x\,dx$

(c) $\int_0^1 x^2e^{-x}\,dx$

6. Evaluate the following integrals: $x^\alpha \sin x$ or $x^\alpha \cos x$ where α would be the times of using by parts

   (a) $\int_0^{\frac{\pi}{2}} x\sin x\,dx$

   (b) $\int_0^{\frac{\pi}{2}} x^2\cos x\,dx$

   (c) $\int_0^{\frac{\pi}{2}} x\sin^2 x\,dx$

   (d) $a_n = \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)\,dx$

7. Evaluate the following integrals: $x^\alpha \arcsin x$ or $x^\alpha \arccos x$ where α would be the times of using by parts

   (a) $\int_0^{\frac{\pi}{4}} x\arcsin x\,dx$

   (b) $\int_0^{\frac{\pi}{4}} x\arccos x\,dx$

8. Evaluate the following integrals: $e^{ax}\sin(bx)$ or $e^{ax}\cos(bx)$

   (a) $\int_0^{\pi} e^{2x}\cos x\,dx$

   (b) $\int_0^{\pi} e^{ax}\sin(bx)\,dx$

   (c) $\int_0^{\pi} e^{ax}\cos(bx)\,dx$

9. Evaluate the following integrals

   (a) $\int_0^{\pi} \cos(\ln x)\,dx$

   (b) $\int_0^{\frac{\pi}{4}} \sec^3 x\,dx$

10. Evaluate $\int_0^2 xf(x)\,dx$ where $f(x) = \int_x^2 \frac{1}{\sqrt{1+t^3}}\,dt$.

11. Consider the integral $I_n = \displaystyle\int_0^1 t^n e^t \, dt$.
    (a) Prove the reduction formula $I_n = e - nI_{n-1}$
    (b) Find I_4 by I_0

12. Consider the integral $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \cos^n x \, dx$.
    (a) Prove the reduction formula $I_n = \dfrac{n-1}{n} I_{n-2}$.
    (b) Find I_n by I_0 and I_1.

13. Consider the integral $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x \, dx$.
    (a) Prove the reduction formula $I_n = \dfrac{n-1}{n} I_{n-2}$.
    (b) Find I_n by I_0 and I_1.

- **Integral by Partial Fraction**

11. Find the following integrals
    (a) $\displaystyle\int_0^1 \frac{1}{x^2+x+1} \, dx$
    (b) $\displaystyle\int_0^1 \frac{x+2}{x^2+x+1} \, dx$
    (c) $\displaystyle\int_5^6 \frac{5x+6}{x^2-x-2} \, dx$
    (d) $\displaystyle\int_5^6 \frac{x^2-1}{x^4+1} \, dx$
    (e) $\displaystyle\int_1^2 \frac{1}{x(x^6+2)} \, dx$
    (f) $\displaystyle\int_5^6 \frac{-x^2+2x+1}{(x-1)^2(x^2+1)} \, dx$

- **Integral by Combination of Techniques**

1. Evaluate the following integrals
    (a) $\displaystyle\int_0^1 \frac{1}{\sqrt{1+x^2}} \, dx$
    (b) $\displaystyle\int_0^1 \sqrt{1+x^2} \, dx$
    (c) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1+x^2}} \, dx$

(d) $\displaystyle\int_e^{\pi} \frac{6e^{2x}}{e^{2x}+2e^x-3}\,dx$

(e) $\displaystyle\int_0^{1} \frac{-3}{1+2e^x-e^{-x}}\,dx$

(f) $\displaystyle\int_0^{1} \frac{x^2}{\sqrt{4x-x^2}}\,dx$

(g) $\displaystyle\int_0^{1} \frac{1}{x+2\sqrt{x+3}}\,dx$

(h) $\displaystyle\int_0^{1} \frac{1}{x+2\sqrt{x-10}}\,dx$

(i) $\displaystyle\int_0^{L} \frac{\sqrt{L^2-x^2}}{x}\,dx$

Problem 6: Explore the general properties of integral

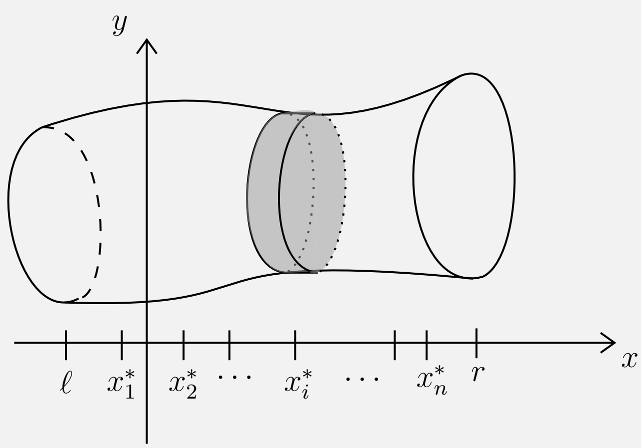
1. Prove that $\displaystyle\int_\ell^r f(t+a)\,dt = \int_{\ell+a}^{r+a} f(t)\,dt$ given $f(t)$ is integrable on $\mathbb{R}$.
2. Prove the following equations:
   (a) $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-t)\,dt$ where $f(x)$ is integrable and $x+t=a+b$.
   (b) $\displaystyle\int_a^b f(x)\,dx = \int_0^1 f((b-a)t+a)\,dt$ where $f(x)$ is integrable and $x=(b-a)t+a$.
3. Prove $\displaystyle\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} f(\sin x)\,dx$.
4. Prove $\displaystyle\int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$
5. Prove $\displaystyle\int_0^r \left(\int_0^s f(t)\,dt\right)\,ds = \int_0^r (r-u)f(u)\,du$ where $f(x)$ is continuous everywhere.

Application of Integral Calculus

1. **Volume**

   (a) **Definition**

   We want to find the volume of a three-dimensional solid with arbitrary shape. Inspired by the idea of idea, we are going to split the solid infinitely small pieces and add all them up.

   > **Definition 3.** Consider the three-dimensional solid below. We define the volume of it by four steps:
   >
   > 1° Partition the interval $[\ell, r]$ into n subintervals $[x_{i-1}, x_i]$ with equal width $\Delta x = x_i - x_{i-1} = \dfrac{r-\ell}{n}$
   >
   > 2° Pick up all the sample (representative) points $x_i^* \in [x_{i-1}, x_i]$
   >
   > 3° Add up all the small rectangles $\displaystyle\sum_{i=1}^{n} A(x_i^*)\Delta x$ where $A(x)$ is area of the cross-section at point x
   >
   > 4° Take the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} A(x_i^*)\Delta x$
   >
   > 
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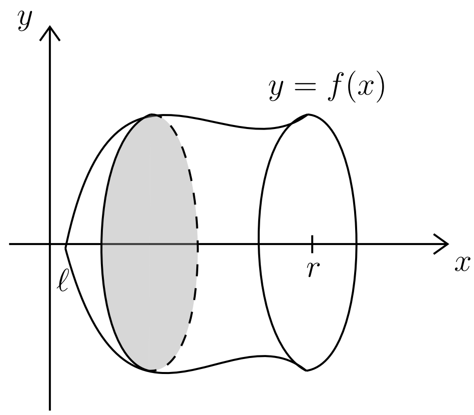
   >
   > Then the volume of a solid on the interval $[\ell, r]$ having cross-section area $A(x)$ at position x is equal to $V = \displaystyle\int_{\ell}^{r} A(t)\, dt$ provided this integral exists.

   (b) **Method**

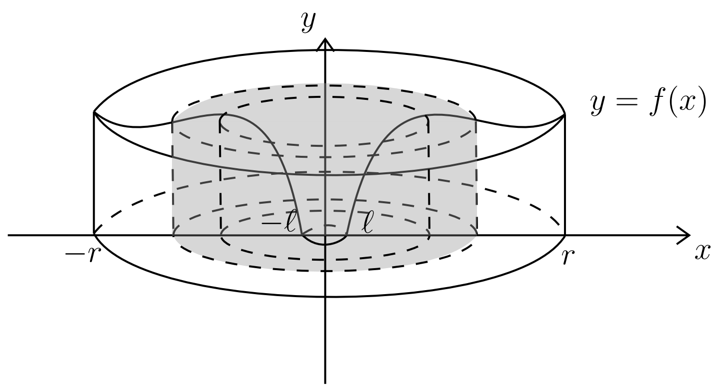
   For the solid of revolution, there are two methods to find the volume: by disks and by shells. They differ from the way we split the solid.

   Let's look at the disk method first. Consider the solid that the region under $y = f(x)$ ($x \in [\ell, r]$) and above $x = 0$ rotated about x-axis.



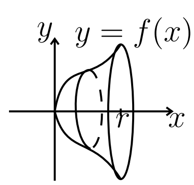
The volume of one piece at x is $dV = A(x)\,dx = \pi f^2(x)\,dx$. Then the volume of the solid is $V = \int_\ell^r \pi f^2(x)\,dx$.

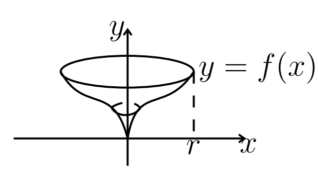
Then let's look at the method of (cylindrical) shell. Consider the solid that the region under $y = f(x)$ ($x \in [\ell, r]$) and above $x = 0$ rotated along y-axis.



Then the volume of one piece of shell at x with height $f(x)$ and thickness dx is $dV = 2\pi x f(x)$. Then volume of the solid is $V = \int_\ell^r 2\pi x f(x)\,dx$.

Then we are going to compare these two methods in different situations. Let $y = f(x)$ ($x \in [0, r]$) rotate around x-axis and y-axis.



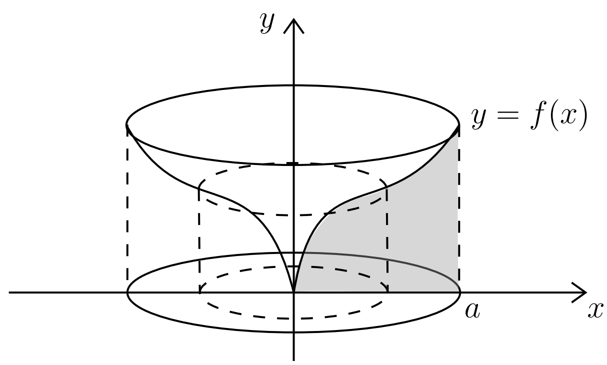


By Disk	$V = \int_0^r \pi f^2(x)\,dx$	$V = \int_0^r 2\pi x[f(r) - f(x)]\,dx$
By Shell	$V = \int_0^{f(r)} \pi[f^{-1}(y)]^2\,dy$	$V = \int_0^{f(r)} 2\pi f(x)(r - x)\,dx$

If f^{-1} cannot be found, then we should use by disk to calculate the rotation along y-axis case. In practice, we usually pick up the easiest way to go.

(c) **Equivalence between Two Methods**
It's hard to show that two methods are equivalent in general. But we could find a special case to prove they are equivalent.

(From the note by Dr. Fok-shuen Leung) Let $f(x)$ be a continuously differentiable function which passes through the original and is strictly increasing. Let R be the region enclosed by the x-axis and $y = f(x)$ from $x = 0$ to $x = a$. Let S denote the solid obtained by rotating R about the y-axis.



Then by disk, we have the volume to be

$$V_d = \int_0^{f(a)} \pi(a^2 - [f^{-1}(y)]^2)\, dy$$

Then by shell, we have the volume to be

$$V_s = \int_0^a 2\pi x f(x)\, dx$$

We could substitute to compare two volumes:

$$\begin{aligned}
V_r \overset{y=f(x)}{\underset{x=f^{-1}(y)}{=\!=\!=}} & \int_0^a \underbrace{\pi(a^2 - x^2) f'(x)}_{\text{continuous so integrable}}\; dt \\
&= \int_0^a \pi(a^2 - x^2)\, d[f(x)] \\
&= \pi(a^2 - x^2) f(x)\Big|_0^a + \int_0^a 2\pi x f(x)\, dx \\
&= \int_0^a 2\pi x f(x)\, dx \\
&= V_c
\end{aligned}$$

2. **Work**

(a) **Definition**

Work is defined as the product of the force and distance if the force is applied constantly. However, if the force is not applied constantly, we have to split the process into pieces and in every small piece the force is constant.

> **Definition 4.** Suppose an object moves along the x-axis from $x = \ell$ to $x = r$, with a force of $F(x)$ acting in the some direction on the object at any point x. We define the work of it by four steps:
>
> 1° Partition the interval $[\ell, r]$ into n subintervals $[x_{i-1}, x_i]$ with equal width $\Delta x = x_i - x_{i-1} = \dfrac{r-\ell}{n}$
>
> 2° Pick up all the sample (representative) points $x_i^* \in [x_{i-1}, x_i]$
>
> 3° Add up all the small rectangles $\displaystyle\sum_{i=1}^{n} A(x_i^*)\Delta x$ where $A(x)$ is area of the cross-section at point x
>
> 4° Take the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} A(x_i^*)\Delta x$
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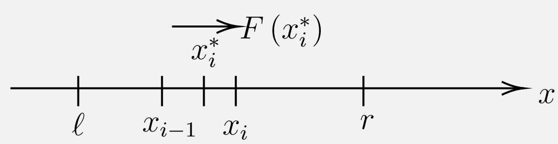
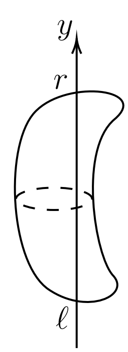
> 
>
>
> Then the work done on the object is equal to $W = \int_r^\ell F(x)\,dx$ provided the integral exists.

(b) **Method**

In general we could just calculate the work by definition

$$W = \int_\ell^r F(x)\,dx$$

However, if we have a continuous solid, we could evaluate the work alternatively - instead of splitting the process, we split the volume. Consider a continuous solid along y-axis bounded by $[\ell, r]$.



We assume that at the same height y, the forces applied on every point are the same. Then we have

$$W = \int_{\ell}^{r} h(y)G(y)\,dV = \int_{\ell}^{r} h(y)G(y)S(y)\,dy$$

where $h(y)$ is the distance of the points at height y have to move, $G(y)$ is factor that force by volume and $S(y)$ is the area at height y.

3. **Differential Equation**

(a) **Definition**

Many quantities grow or decay with a rate proportional to themselves. For example, a colony of bacteria double its size every hour. We could write that in equation

$$\frac{dy}{dt} = y$$

It is related to derivatives and we call it differential equation.

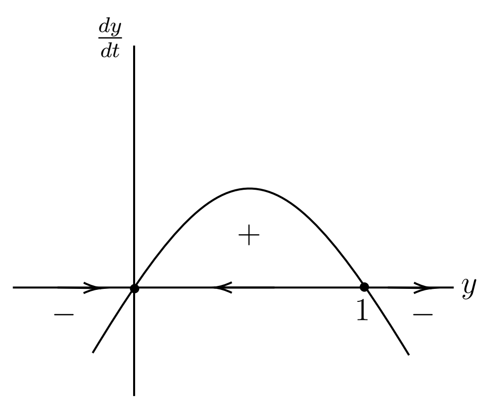
> **Definition 5.** (i) *Differential equations* are equations relating functions and the derivatives. (ii) The *solutions* of the differential equations are functions that satisfy the differential equation after substituting in. *Special solution* has no arbitrary constant and *general solution* has no linear dependent constants whose number is equal to the order of the differential equation. (iii) If $y_0 = f(t_0)$ specify the solution containing (t_0, y_0), we say $y_0 = f(t_0)$ is the *initial condition.* The number of initial conditions to specify a special solution equals to the order of the linear equation.

(b) **Direction Fields and Phase Portraits**

We want to visualize the differential equation to help understand it. Direction fields and phase portraits are powerful tools that help us do so. For example, consider

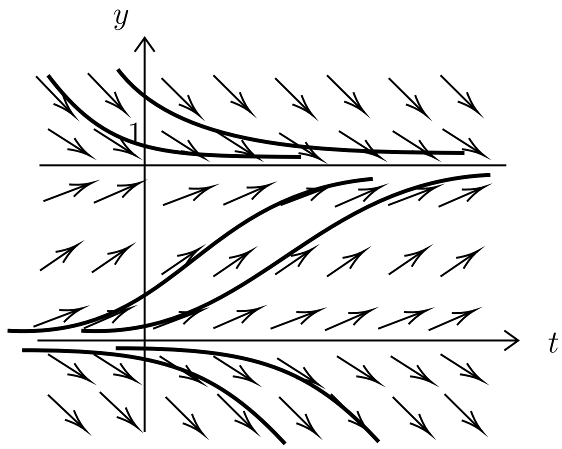
$$\frac{dy}{dt} = (1-y)y$$

We have the phase portrait



When $y < 0$ or $y > 1$, $\dfrac{dy}{dt} < 0$ and y is decreasing. When $0 < y < 1$, $\dfrac{dy}{dt} > 0$ and y is increasing. When $y = 0$ or $y = 1$, $\dfrac{dy}{dt} = 0$ and y remains steady.

Based on that, we could add more information to draw a direction field.



(c) **Separation of Variables**

If we have

$$\frac{dy}{dt} = f(t, y) = \varphi_1(t)\varphi_2(y)$$

Then we could separate variables of t and y.

- Suppose $\varphi_2(y) = 0$ which solves $y = C$. Then we have to plug it into the equation to check whether it is a solution.
- Suppose $\varphi_1(y) \neq 0$. Then we could have

$$\int \frac{dy}{\varphi_2(y)} = \int \varphi_1(t)\, dt$$

  which gives $y = g_C(x)$

Then we could combine or summary two solutions to find out the final solution.

Let's look at an example.

Example 14. Solve $\dfrac{dy}{dt} = ty$.

Suppose $y = 0$. Plug in, we could find it is the solution.

Suppose $y \neq 0$. Separate variables,

$$\int \frac{dy}{y} = \int t\, dt$$

Then we have

$$\ln|y| = \frac{1}{2}t^2 + C_0$$

which solves
$$y = \pm e^{C_0} e^{\frac{1}{2}t^2} = Ce^{\frac{1}{2}t^2}$$
In summary the solution is
$$y = Ce^{\frac{1}{2}t^2}$$

(d) **Integrating Factors**
If we have the differential equation
$$\frac{dy}{dt} + P(t) = Q(t)$$
we wish to find an integrating factor $I(t)$ such that
$$\underbrace{\frac{dy}{dt}I(t) + P(t)yI(t)}_{=\frac{d}{dt}(yI(t))} = Q(t)I(t)$$
So we have
$$\frac{d}{dt}(I(t)y) = \cancel{\frac{dy}{dt}I(t)} + \frac{dI}{dt}\cancel{y} = \cancel{\frac{dy}{dt}I(t)} + P(t)\cancel{y}I(t)$$
which implies
$$\int \frac{dI}{I} = \int P(t)\,dt$$
Therefore
$$I(t) = \pm e^{\int P(t)\,dt + C}$$
For convenience, we pick $I(t) = e^{\int P(t)}$. So we only have to solve
$$\frac{dy}{dt}(yI(t)) = Q(t)I(t)$$
This turns out to be $\displaystyle\int d(yI(t)) = \int Q(t)I(t)\,dt$. So finally we get
$$y = \frac{\int Q(t)I(t)\,dt + C}{I(t)} = \frac{\int Q(t)e^{\int P(t)}\,dt + C}{e^{\int P(t)}}$$

Let's look at an example.

Example 15. Solve $\dfrac{dy}{dx} - \dfrac{y}{x} = 1$. When $x = 1$, $y = 1$. The general solution is
$$y = \frac{\int e^{\int -\frac{1}{x}\,dx} + C}{e^{\int -\frac{1}{x}\,dx}} = \frac{\ln|x| + C}{\left|\frac{1}{x}\right|}$$
Plug in the initial condition, we have $C = 1$. Then the solution is
$$y = |x| \ln|x|$$

Problem Solving III

Evaluate the volume and work; Sketch the direction field, solve differential equations with application

Problem 7: Evaluate the volume and work

1. Let R be the bounded region between the two curves $y = \sqrt[4]{x}$ and $y = x$. Find the volume of the solid that is generated by rotating the region R about the vertical line $x = 1$.
2. Find the volume of the sphere $x^2 + y^2 + z^2 = r^2$ $(r > 0)$.
3. A bucket with a mass of 30 kg when filled with sand needs to be lifted to the top of a 20 meter tall building. We have a rope that has a mass of 0.2 kg/m that takes 1 meter to secure to the bucket. Once the bucket reaches the top of the building, it has a mass of only 19 kg because there is a hole in the bottom and sand was leaking out at a constant rate while it was being lifted to the top of the building. Find the work done lifting the bucket, sand and rope to the top of the building.

Problem 8: Sketch the direction field, solve differential equations with application

- **Sketch direction fields and solve differential equations**
  1. Sketch the direction fields of the following differential equation.
     (a) $\dfrac{dy}{dx} = x - y$
     (b) $\dfrac{dy}{dx} = \sin(x)\sin(y)$
  2. Solve the following differential equations
     (a) $\dfrac{dy}{dx} = x - y$, $y(1) = 1$
     (b) $\dfrac{dy}{dx} = \sin(x)\sin(y)$, $y(0) = 1$
     (c) $\dfrac{dy}{dx} = (e^x + x)y^2$
     (d) $\dfrac{dy}{dx} - x^2 y = e^x$
     (e) $x\dfrac{dy}{dx} + y + xy^2 = 0$
  3. Solve the following integral equations
     (a) $f(x) = 1 + \displaystyle\int_1^x t f(t)\, dt$
     (b) $f(x) = 1 + \displaystyle\int_1^x t(1 - f(t))\, dt$
     (c) $f(x) = 1 + \displaystyle\int_0^x \frac{f^2(t)}{1 + t^2}\, dt$

- **Application of differential equation**

4. A tank has pure water flowing into it at 10 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 L/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 L of water.
5. A tank has pure water flowing into it at 10 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 L/min. Initially, the tank contains 10 kg of salt in 100 L of water.
6. Find an infinite number of curves that intersect orthogonally all ellipse of the form $x^2 + 2y^2 = a^2$.
7. Find the orthogonal trajectories of the families of the curves defined by $y^2 = kx^3$.
8. Find the orthogonal trajectories of the families of the curves defined by $y^m = kx^n$.

Series, Power Series and Approximation

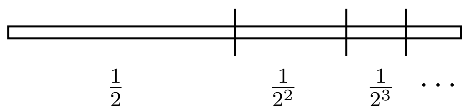
1. **Series**

   (a) **Definition**

   Recall the definition of the integral,

$$\int_{\ell}^{r} f(x)\,dx = \sum_{n=1}^{\infty} f(x_i^*)\Delta x$$

which is the sum of infinite elements. In fact, this is very common. Let's look at an example from ancient China. In *Zhuangzi*, it is noted that "Cutting off a foot-long club by half every day will not exhaust it any day."



We could see the lengths cut off are

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots, \frac{1}{2^n}, \cdots$$

In other words, on the nth day, the length of club cut off is

$$a_n = \frac{1}{2^n}$$

If we sum them up, the total length cut off up till nth day is

$$S_n = \sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} < 1$$

Therefore, Zhuangzi is right that we could never exhaust it. We could also take the limit,

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \lim_{n\to\infty} S_n = 1$$

This shows that the total length cut off is approaching to the initial length of the club but could not actually reach it.

Then we wish to make formal definitions of S_n and the infinite sum.

> **Definition 6.** A *series* is a formal infinite sum of elements. The *partial sum* of the series $\displaystyle\sum_{n=1}^{\infty} a_n$ (alternatively, it could be denoted as $\displaystyle\sum_{n\geq 1} a_n$) is $S_n = a_1 + a_2 + \cdots + a_n$. The series *converges* to L if the limit of its partial sum converges to L $\displaystyle\lim_{n\to\infty} S_n = L$. The series diverges if the limit of its partial sum diverges.

(b) **Two Basic Series**

- **Geometric Series:** $\sum_{n=1}^{\infty} ar^n \ (a, r \neq 0)$

  If $|r| \neq 1$, the partial sum is

  $$S_n = \frac{a(1-r^n)}{1-r}$$

  Then we take the limit of S_n

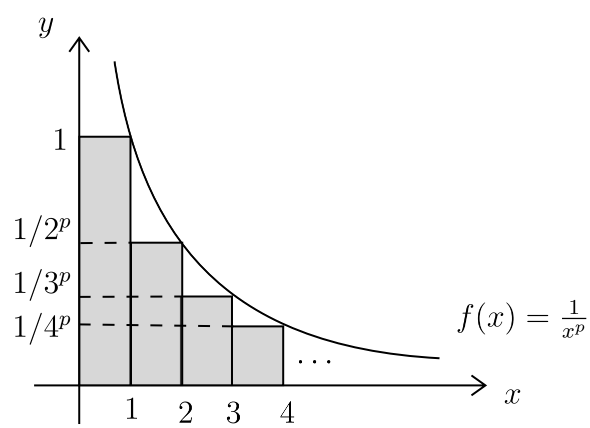
  $$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \begin{cases} \dfrac{a}{1-r} \text{ (converges)} & |r| < 1 \\ \text{diverges} & |r| > 1 \end{cases}$$

  If $r = 1$, the partial sum is $S_n = na$ and the limit $\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = +\infty$ diverges. If $r = -1$, the partial sum is $S_n = -\frac{a}{2} + (-1)^n \frac{a}{2}$ and the limit $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(-\frac{a}{2} + (-1)^n \frac{a}{2}\right)$ diverges. Therefore the series

  $$\sum_{n=1}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r} \text{ (converges)} & |r| < 1 \\ \text{diverges} & |r| \geq 1 \end{cases}$$

- **p-Series:** $\sum_{n=1}^{\infty} \frac{1}{n^p} (p > 0)$

  If $p > 1$, we have the partial sum $S_n \leq \sum_{n=1}^{\infty} \frac{1}{n^p}$ and $S_n \leq S_{n+1}$ since $\frac{1}{n^p} > 0$. And by comparing the area



  we know

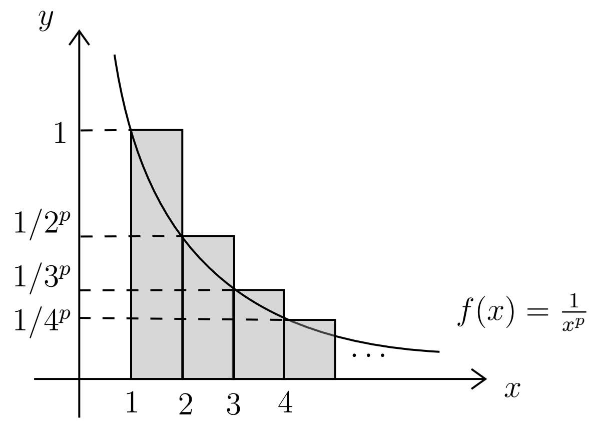
  $$S_n \leq \sum_{n=1}^{\infty} \frac{1}{n^p} \leq 1 + \int_1^{\infty} \frac{1}{x^p}\, dx$$

Since $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{a\to\infty} \frac{1}{1-p}x^{1-p}\Big|_1^a = -\frac{1}{1-p}$, we know

$$S_n \le \frac{p}{p-1}$$

Therefore S_n is monotone and bounded then converges. Therefore $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

If $p \ge 1$, we have the partial sum $S_n \le \sum_{n=1}^{\infty} \frac{1}{n^p}$ and $S_n \le S_{n+1}$ since $\frac{1}{n^p} > 0$. And by comparing the area



we have

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \ge \int_1^\infty \frac{1}{x^p}\,dx$$

Since $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{a\to\infty} \frac{1}{1-p}x^{1-p}\Big|_1^a = +\infty$ $(p < 1)$ or $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{a\to\infty} \ln(x)|_1^a = +\infty$ $(p = 1)$ diverge, we know $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

In summary we have

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \le 1 \end{cases}$$

When $p = 1$, we say $\sum_{n=1}^{\infty} \frac{1}{n}$ is *harmonic series*. $\sum_{n=1}^{\infty} \frac{1}{n}$ for sure diverges and we are going two more proofs.

Proof 1: We group up the series

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \cdots \\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{16} + \cdots \\
&= 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty
\end{aligned}$$

Proof 2: Assume the harmonic series converges to L. Then we have

$$\begin{aligned}
L &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + \cdots \\
&= 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = L + \frac{1}{2}
\end{aligned}$$

which gives a contradiction.

(c) **Basic Properties**

i. $\displaystyle\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$ if $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converge.

Proof. Let $A_n = a_1 + a_2 + \cdots + a_n$, $B_n = b_1 + b_2 + \cdots + b_n$ and $S_n = (a_1 \pm b_1) + \cdots + (a_n \pm b_n)$. Then we have

$$S_n = A_n \pm B_n$$

Then we take the limit

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (A_n \pm B_n) = \lim_{n\to\infty} A_n \pm \lim_{n\to\infty} B_n$$

which is

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

ii. $\displaystyle\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n$ if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. $\displaystyle\sum_{n=1}^{\infty} ka_n$ diverges if $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Let $S_n = a_1 + \cdots + a_n$ and $S'_n = ka_1 + \cdots + ka_n = kS_n$. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges to S, we know

$$\lim_{n\to\infty} S'_n = \lim_{n\to\infty} kS_n = k \lim_{n\to\infty} S_n = kS$$

Therefore $\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n$. If $\sum_{n=1}^{\infty} a_n$ diverges, we know $\lim_{n\to\infty} S_n' = \lim_{n\to\infty} kS_n = k\lim_{n\to\infty} S_n$ also diverges and then $\sum_{n=1}^{\infty} ka_n$ also diverges.

iii. $\sum_{n=1}^{\infty} a_n$ converges if and only if, $\sum_{n=k}^{\infty} a_n$ converges for any positive integer k.

Proof. Let $a_1 + \cdots + a_{k-1} = M$. Let $S_n = a_1 + \cdots + a_n$ and $S_n' = a_k + \cdots + a_n$ $(n \geq k)$. If $\sum_{n=1}^{\infty} a_n$ converges to S, we have

$$\sum_{n=k}^{\infty} a_n = \lim_{n\to\infty} S_n' = \lim_{n\to\infty} (S_n - M) = S - M$$

also converges. If $\sum_{n=k}^{\infty} a_n$ converges to S, we have

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (S_n' + M) = S + M$$

also converges.

iv. **Divergence Test:** If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$.

Example 16. Consider whether $\sum_{n=1}^{\infty} (-1)^n$ converges. Since $\lim_{n\to\infty} (-1)^n$ diverges, by divergence test, $\sum_{n=1}^{\infty} (-1)^n$ diverges.

Note: It shows necessary condition of series to converge: a_n should finally converges 0. But converging to 0 does not guarantee the convergence of the series, the speed of a_n approaching to 0 also matters.

(d) **Convergence Tests**

- **To Compare:** $\left.\begin{aligned} a_n > 0 \Rightarrow S_n \nearrow \\ S_n \leq M \end{aligned}\right\} \Rightarrow S_n$ converges

i. **Comparison Test:** Let $a_n, b_n \geq 0$. If $a_n \geq b_n$ and $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} b_n$ converges. If $a_n \leq b_n$ and $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. Let S_n and T_n to be the partial sum of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

If $a_n \geq b_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges. We know $S_n \geq T_n$. Let $\lim_{n\to\infty} S_n = S$. Then S is the upper bound of S_n since $S_n - S_{n-1} = a_n \geq 0$. To $S \geq S_n \geq T_n$. Since T_n is increasing by $b_n \geq 0$, $\lim_{n\to\infty} T_n$ exists. Therefore, $\sum_{n=1}^{\infty} b_n$ converges.

If $b_n \geq a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges. We have $S_n \leq T_n$. Since $\lim_{n\to\infty} S_n = +\infty$, $\lim_{n\to\infty} T_n = +\infty$. So $\sum_{n=1}^{\infty} b_n$ diverges.

Example 17. We have proved the harmonic series diverges. Based on this, we could prove $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p < 1)$ diverges without integral. Since $\frac{1}{n^p} > \frac{1}{n} > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by comparison test, we have $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

ii. **Limit Comparison Test:** If $a_n \geq 0$, $b_n \geq 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then both series converge or diverge.

Proof. Let $\varepsilon = \frac{L}{2} > 0$, there exists $n > N$, $\left|\frac{a_n}{b_n} - L\right| < \frac{L}{2}$ then $\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$. This yields

$$b_n < \frac{2}{L} a_n,\ a_n < \frac{3L}{2} b_n$$

By comparison test, $\sum_{n=N}^{\infty} a_n$ and $\sum_{n=N}^{\infty} b_n$ both converges. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converges.

Corollary:

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

  Proof. For sufficient large n, $a_n < \varepsilon b_n$ with $\varepsilon > 0$. By comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

  Proof. For sufficient large n, $a_n > M b_n$, by comparison test, $\sum_{n=1}^{\infty} a_n$ diverges.

Example 18. Consider whether $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^4+n+7}}$ converges. We need another series $\sum_{n=1}^{\infty} \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{\dfrac{n+1}{\sqrt{n^4+n+7}}}{\dfrac{1}{n}} = \lim_{n\to\infty} \sqrt{\frac{n^4+2n^2+1}{n^4+n+7}} = 1$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by limit comparison test, we know $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^4+n+7}}$ diverges.

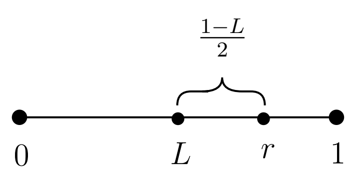
Note: Basically, these two tests, besides comparing whether two series converge to 0, also compare how quickly they approach to 0, which helps us determine the convergence of two series.

- **To Self-Test**

iii. **Ratio Test:** If $a_n > 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, Then we know

$$\sum_{n=1}^{\infty} a_n \begin{cases} \text{converges} & L < 1 \\ \text{diverges} & L > 1 \\ \text{exclusive} & L = 1 \end{cases}$$

Proof. If $L < 1$, let $\varepsilon = \frac{1-L}{2} > 0$. There exists $N > 0$ such that when $n > N$, $\left|\frac{a_{n+1}}{a_n} - L\right| < \frac{1-L}{2}$. Then we get $\frac{a_{n+1}}{a_n} < \frac{1-L}{2} + L \equiv r < 1$.



Then we have

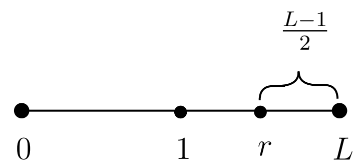
$$\begin{aligned} a_{N+1} &< ra_N \\ a_{N+2} &< ra_{N+1} < r^2a_{N+2} \\ a_{N+3} &< ra_{N+2} < r^2a_{N+1} < r^3a_{N+1} \\ &\cdots \end{aligned}$$

Therefore we have

$$a_N + a_{N+1} + a_{N+2} + \cdots < a_N + ra_N + r^2a_N + \cdots = a_N(1 + r + r^2 + \cdots)$$

Since $\sum_{n=1}^{\infty} a_n r^{n-N}$ converges by $r < 1$, by comparison test, $\sum_{n=N}^{\infty} a_n$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges.

If $L > 1$, let $\varepsilon = \frac{L-1}{2} > 0$. There exists $N > 0$ such that when $n > N$, $\left|\frac{a_{n+1}}{a_n} - L\right| < \frac{L-1}{2}$. Then $\frac{a_{N+1}}{a_N} > \frac{1-L}{2} + L \equiv r > 1$.



For the same reason

$$a_N + a_{N+1} + a_{N+2} + \cdots > a_N + a_N r + a_N r^2 + \cdots$$

Since $\sum_{n=1}^{\infty} a_n r^{n-N}$ diverges by $r > 1$, by comparison test, $\sum_{n=N}^{\infty} a_n$ diverges. Therefore $\sum_{n=1}^{\infty} a_n$ diverges.

Example 19. Consider whether $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \frac{1}{2} < 1$$

By ratio test, we know $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

iv. **Root Test:** If $a_n \geq 0$ and $\lim_{n\to\infty} \sqrt[n]{a_n} = L$, then we know

$$\sum_{n=1}^{\infty} a_n \begin{cases} \text{converges} & L < 1 \\ \text{diverges} & L > 1 \\ \text{exclusive} & L = 1 \end{cases}$$

Proof. If $L < 1$,by applying the same trick in the proof of ratio test, we could prove there exists $N > 0$ such that when $n > N$, $a_n < r^n$ $(r < 1)$. By comparison test, $\sum_{n=N}^{\infty} a_n$ converges and then $\sum_{n=1}^{\infty} a_n$ converges. If $L > 1$,by applying the same trick in the proof of ratio test, we could prove there exists $N > 0$ such that when $n > N$, $a_n > r^n$ $(r > 1)$. By comparison test, $\sum_{n=N}^{\infty} a_n$ diverges and then $\sum_{n=1}^{\infty} a_n$

diverges.

Example 20. Consider whether $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^n$ converges. Since

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{1}{n+1}=0<1$$

by root test, we know $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^n$ converges.

v. **Raabe–Duhamel's Test:** If $a_n>0$ and $\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right)=L$, then we know

$$\sum_{n=1}^{\infty}a_n\begin{cases}\text{converges} & L>1\\ \text{diverges} & L>1\\ \text{exclusive} & L=1\end{cases}$$

Proof. If $L>1$, by applying the same trick in the proof of ratio test, there exists $N>0$ such that when $n\geq N$ $n\left(1-\frac{a_{n+1}}{a_n}\right)>r>1$. Then we have

$$ra_n<na_n-na_{n+1}$$

Subtract a_n on both sides

$$(r-1)a_n<(n-1)a_n-na_{n+1}$$

Then we have

$$\begin{array}{lcl}(r-1)a_N & < & (N-1)a_N-Na_{N+1}\\ (r-1)a_{N+1} & < & Na_{N+1}-(N+1)a_{N+1}\\ & \vdots & \\ (r-1)a_n & < & (n-1)a_n-na_{n+1}\end{array}$$

which implies

$$(r-1)(a_N+a_{N+1}+\cdots+a_n)<(N-1)a_N-na_{n+1}<(N-1)a_N$$

Then the partial sum

$$S_n=a_1+\cdots+a_{N-1}+a_N+\cdots+a_n<a_1+\cdots+a_{N-1}+a_N+\frac{N-1}{r-1}a_N$$

Since $a_n>0$, S_n is increasing. Since S_n is bounded and monotone, S_n converges and then $\sum_{n=1}^{\infty}a_n$ converges.

If $L < 1$, by applying the same trick, there exists $N > 2 > 0$ such that when $n \geq N$ $n\left(1 - \frac{a_{n+1}}{a_n}\right) < 1$. This could be simplified as

$$a_{n+1} > (n-1)a_n$$

Therefore for $n > N$

$$(n-1)a_n > (n-2)a_{n-1} > \cdots > (N-1)a_N > a_N$$

Then we have $a_n > \frac{a_N}{n-1}$. Since $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{a_N}{n-1}} = \frac{1}{a_N} > 0$, by limit comparison test, $\sum_{n=N}^{\infty} \frac{a_N}{n-1}$ diverges as the harmonic series. By comparison test, $\sum_{n=N}^{\infty} a_n$ diverges and then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 21. We could prove $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p > 1)$ converges and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p < 1)$ diverges in another way. By binomial theorem (which will be proved in the next sections),

$$(n+1)^p = n^p + pn^{p-1} + O(n^{p-1})$$

Then we know the limit

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \frac{n(\cancel{n^p} + pn^{p-1} + O(n^{p-1}) - \cancel{n^p})}{(n+1)^p} = \lim_{n\to\infty} \frac{pn^p}{(n+1)^p} = p$$

Therefore if $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges and if $p < 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Note: The root test (r^n), ratio test (r^n) and Raabe–Duhamel's test ($\frac{1}{n^p}$) all check how fast a_n approaches to 0 but the requirement to pass (showing convergence) the Raabe–Duhamel's test is lower, which needs a slower speed approaching (finer difference).

vi. **Integral Test:** Let $a_n = f(n)$, where $f(x)$ is continuous, positive, and non-increasing for $x \geq 1$. Then $\int_1^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.

Proof. If $\int_1^{\infty} f(x)\,dx$ converges to M. Then when $n \geq 2$, for $x \in [n-1, n)$, we have $f(x) > f(n) = a_n$ since $f(x)$ is non-increasing. Take the integral on both sides (integrable since $f(x)$ is continuous)

$$\int_{n-1}^{n} f(x)\,dx > a_n$$

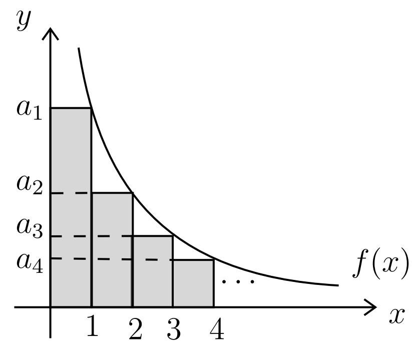
Add them up, we have

$$\int_1^{\infty} f(x)\,dx > a_2 + \cdots + a_n + \cdots$$

Then we know the partial sum

$$S_n < a_1 + \int_1^{\infty} f(x)\,dx = a_1 + M$$

which is bounded. Besides, since $a_n = f(n) > 0$, S_n is increasing. Then S_n converges and then $\sum_{n=1}^{\infty} a_n$ converges.



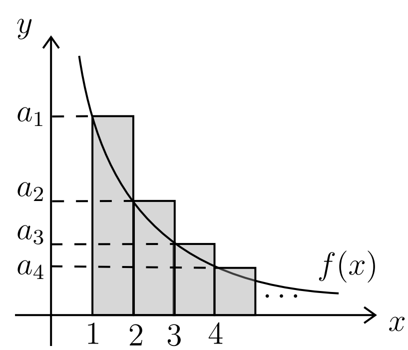
If $\int_1^{\infty} f(x)\,dx$ diverges to $+\infty$. Then when $n \geq 1$, for $x \in (n, n+1]$, we have $f(x) < f(n) = a_n$ since $f(x)$ is non-increasing. Take the integral on both sides (integrable since $f(x)$ is continuous)

$$\int_n^{n+1} f(x)\,dx < a_n$$

Add them up, we have

$$\sum_{n=1}^{\infty} = a_1 + \cdots + a_n + \cdots > \int_1^{\infty} f(x)\,dx = +\infty$$

also diverges.



Example 22. Consider whether $\sum_{n=10}^{\infty} \frac{1}{n\ln(n)}$. The corresponding function is $f(x) = \frac{1}{x\ln(x)}$. Then $f'(x) = -\frac{1+\ln(x)}{(x\ln(x))^2} < 0$ and $f(x)$ is non-increasing when $x > 10$. Besides, $f(x)$ is positive, continuous when $x \geq 10$. Since

$$\int_{10}^{\infty} \frac{1}{x\ln(x)}\, dx = \lim_{r\to\infty} \ln(\ln(x))|_{10}^{r} = +\infty$$

diverges, by integral test, $\sum_{n=10}^{\infty} \frac{1}{n\ln(n)}$ diverges.

- **To Explore Negative**

vii. **Alternating Series Test:** For the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ where $a_n > 0$ for any $n \geq 1$. If $a_n \geq a_{n+1}$ for $n \geq 1$ and $\lim_{n\to\infty} a_n = 0$, then the series converges.

Proof. We could group the partial sum S_{2n} in two different ways.

$$S_{2n} = (\underbrace{a_1 - a_2}_{\geq 0}) + \cdots + (\underbrace{a_{2n-1} - a_{2n}}_{\geq 0})$$

Then $S_2 \leq S_4 \leq \cdots \leq S_{2n}$, which means S_{2n} is increasing. Also, we have

$$S_{2n} = a_1 - (\underbrace{a_2 - a_3}_{\leq 0}) - \cdots - (\underbrace{a_{2n-2} - a_{2n-1}}_{\leq 0}) - a_{2n} \leq a_1$$

is bounded. Therefore,

$$\lim_{n\to\infty} S_{2n}$$

converges. Since $S_{2n+1} = S_{2n} + a_{2n+1}$, we have

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \underbrace{\lim_{n\to\infty} a_{2n+1}}_{=0} = \lim_{n\to\infty} S_{2n}$$

also converges. Therefore $\lim_{n\to\infty} S_n$ converges and then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

Example 23. Consider the whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. Since $\frac{1}{n} > 0$ decrease and $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Notice the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right|$ diverges. From this example, we could see adding absolute value makes the series more easier to diverge. In fact, this is a general result about the absolute value of of a series.

Claim: If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Proof 1. We have

$$a_n = \frac{|a_n| + a_n}{2} - \frac{|a_n| - a_n}{2}$$

Since

$$0 \le \frac{|a_n| + a_n}{2} \le |a_n|,\ 0 \le \frac{|a_n| - a_n}{2} \le |a_n|$$

by comparison test, $\sum_{n=1}^{\infty}\frac{|a_n| + a_n}{2}$ and $\sum_{n=1}^{\infty}\frac{|a_n| - a_n}{2}$ converge. Therefore

$$\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\frac{|a_n| + a_n}{2} + \sum_{n=1}^{\infty}\frac{|a_n| - a_n}{2}$$

converges.

Proof 2. We have

$$0 \le a_n + |a_n| \le 2|a_n|$$

Since $\sum_{n=1}^{\infty}|a_n|$ converges, $\sum_{n=1}^{\infty}2|a_n|$ converges. By comparison test, $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges. Therefore $\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty}|a_n|$ also converges.

The convergence of $\sum_{n=1}^{\infty}|a_n|$ is stronger than the convergence $\sum_{n=1}^{\infty}a_n$. we want to define the stronger convergence as *absolute convergence* and weaker convergence as *conditional convergence*.

Definition 7. If $\sum_{n=1}^{\infty}a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges, then we say $\sum_{n=1}^{\infty}a_n$ *conditionally converges*. If $\sum_{n=1}^{\infty}|a_n|$ converges, then we say $\sum_{n=1}^{\infty}a_n$ absolutely converges.

2. **Power Series**

   (a) **Definition**

   Polynomial is the most common function in practice. When there are infinite terms, we have a power series.

Definition 8. A series

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is a *power series* about c. The constants a_0, a_2, $\ldots$ are the *coefficients of the series*; and c is the *centre of convergence* where the series always converges.

We could treat power series as a function $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ when it converges.

(b) **Radius of Convergence and Interval of Convergence**

We are interested in when the power series converges, i.e. when the power series converges to a function $f(x)$, not diverging to infinity. We have a theorem to help.

Theorem 4. For $\sum_{n=0}^{\infty} a_n(x-c)^n$, exactly on of the following holds:

- the series converges only at $x = c$,
- the series converges everywhere, or,
- the series converges if $|x-c| < R$ and diverges if $|x-c| > R$

Moreover, the convergence in three cases are all absolute.

Proof. First we are going to prove if the series converges at x_0, the the series converges absolutely for any x such that $|x-c| < |x_0 - c|$. Since $\sum_{n=0}^{\infty} a_n(x_0-c)^n$, by divergence test we have $|a_n(x_0-c)^n| \leq a$ for some positive constant a. Thus

$$\sum_{n=0}^{\infty} |a_n(x-c)^n| = \sum_{n=0}^{\infty} |a_n(x_0-c)^n| \left|\frac{x-c}{x_0-c}\right|^n \leq \sum_{n=1} a \left|\frac{x-c}{x_0-c}\right|^n$$

where $|x-c| < |x_0-c|$, converges as a geometry series with $r < 1$. Then there are three possible cases for all the x_0 such that $\sum_{n=0}^{\infty} a_n(x_0-c)^n$ converges:

- $\max\{|x_0-c|\} = 0$
- $\max\{|x_0-c|\} > 0$ is infinite
- $\max\{|x_0-c|\} > 0$ is finite

which imply the three cases in the theorem.

Consider case III, if we find the series converges for $|x-c| < R$ and diverges for $|x-c| > R$, we say R is the series' *radius of convergence*. However, the theorem does not guarantee the convergence of two endpoints $x = c+R$ and $x = c-R$. We have to check them separately. Then we could find $(c-R, c+R)$ with possible endpoints as the *interval of convergence*.

Definition 9. If the series converges for $|x-c| < R$ and diverges for $|x-c| > R$, we say R is the series' *radius of convergence*. The power series converge in *intervals of convergence* centred on the centre of convergence, and diverge elsewhere.

Example 24. Find the radius of convergence and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-x)^n}{4^n n^4}$.

We apply the ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{4} \right| = \left| \frac{x}{4} \right| < 1$$

Then we have $|x| < 4$ and the radius of convergence is $R = 4$. Then we check the endpoints. For $x = 4$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ converges. If $x = -4$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges. Therefore the interval of convergence is $[-4, 4]$.

(c) **Operation on Power Series** (with the same radius of convergence)

i. $\displaystyle\sum_{n=0}^{\infty} (a_n + b_n)(x-c)^n = \sum_{n=0}^{\infty} a_n (x-c)^n + \sum_{n=0}^{\infty} b_n (x-c)^n$

ii. $\displaystyle\left(\sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=0}^{\infty} n a_n x^{n-1}$

iii. $\displaystyle\int_0^x \sum_{n=0}^{\infty} a_n x^n \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^n$

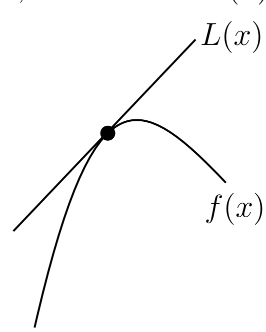
3. **Application: Approximation**

(a) **Linear Approximation**

Recall that, for a function $f(x)$, we could find a tangent line at $x = c$ where $f(x)$ is differentiable.

$$L(x) = f(c) + f'(c)(x-c)$$

We could see when x is close to c, the value of $L(x)$ is very close to $f(x)$.



Therefore, we could treat $L(x)$ as a kind of approximation of $f(x)$, called *linear approximation*.

Definition 10. Let $f(x)$ to be differentiable at $x = c$. The *linear approximation* of $f(x)$ at $x = c$ is

$$f(x) \approx f(c) + f'(c)(x - c)$$

We call the function $L(x) = f(c) + f'(c)(x - c)$ the *linearization* of $f(x)$ at $x = c$.

Example 25. Find the linear approximation of $f(x) = e^x$ at $x = 0$ and make an approximation of $e^{0.1}$.
Solution. $f'(x) = e^x$ and then $f'(0) = f(0) = 1$. Therefore the linear approximation is $f(x) \approx x + 1$. Therefore $e^{0.1} \approx 1.1$.

We are also interested in how accurate the approximation is, i.e. how large the error the approximation is. We have a theorem to help.

Theorem 5. Let $L(x)$ be the linearization of $f(x)$ at c, and let $E(x) = f(x) - L(x)$ be the error. Then there exists a number s between c and x such that

$$E(x) = \frac{f''(s)}{2}(x - c)^2$$

provided $f(x)$ is twice-differentiable in an interval containing c and x.

Proof. Without loss of generality, assume $x > c$. By generalized mean value theorem, we conclude there exists a number a in (c, x) such that

$$\frac{E(x)}{(x-c)^2} = \frac{E(x) - E(c)}{(x-c)^2 - (c-c)^2} = \frac{E'(a)}{2(a-c)}$$

Since $E(t) = f(t) - f(c) - f'(c)(t - c)$, we have $E'(t) = f'(t) - f'(c)$. Therefore

$$\frac{E(x)}{(x-c)^2} = \frac{f'(a) - f'(c)}{2(a-c)}$$

By mean value theorem

$$\frac{E(x)}{(x-c)^2} = \frac{1}{2}f''(s)$$

where $s \in (c, a)$.

Example 26. Estimate the linear approximation of $e^{0.1}$ around $x = 0$.
Solution. The error is

$$E(x) = \frac{1}{2}f''(s)x^2 = \frac{e^s}{2}x^2$$

where $s \in [0, 0.1]$. Since $E(x)$ is increasing, we could estimate the bound

$$|E(x)| \leq \frac{e^{0.1}}{2}(0.1)^2 \approx 0.0055$$

which is small.

(b) **Taylor Series**

i. **Definition**

If possible, we wish to express $f(x)$ in terms of $\sum_{n=0}^{\infty} a_n(x-c)^n$, i.e. to determine the coefficient $\{a_n\}$. In its interval of convergence, we have

$$\begin{array}{ll} f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots & \Rightarrow f(c) = a_0 \\ f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots & \Rightarrow f'(c) = a_1 \\ f''(x) = 2a_2 + 3 \times 2a_3(x-c) + \cdots & \Rightarrow f''(c) = 2a_2 \\ f^{(3)}(x) = 6a_3 + \cdots & \Rightarrow f^{(3)}(c) = 6a_3 \\ \cdots & \cdots \end{array}$$

We could conclude that $a_n = \dfrac{f^{(n)}(c)}{n!}$ and $f(x) = \sum_{n=0}^{\infty} \dfrac{f^n(c)}{n!}(x-c)^n$, which is called Taylor series.

Definition 11. We say the series

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

is the *Taylor series* of $f(x)$ about c. When $c = 0$, we call the series *Maclaurin series* of $f(x)$. The degree n polynomial obtained by truncating the Taylor series of $f(x)$ about c is called the *degree n Taylor polynomial of $f(x)$ about c*.

Let's look at some examples.

Example 27. We want to find the Maclaurin series of $f(x) = e^x$. Since $f^{(n)}(x) = e^x$ for all $n \in \mathbb{N}$, $f^{(n)}(0) = 1$. We have $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$. It's not hard to find the interval of convergence is $(-\infty, +\infty)$.

Example 28. We want to find the Maclaurin series of $f(x) = \dfrac{1}{1-x}$. Recall the geometry series, $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$. We will also prove that by derivative. We have

$$f^{(n)}(x) = \frac{(-1)^{n+1} n!}{(x-1)^{n+1}}$$

and then

$$f^{(n)}(0) = n!$$

Therefore the series is $\sum_{n=0}^{\infty} x^n$. It's not hard to find the interval of convergence is $(-1, 1)$.

Example 29. We want to find the Maclaurin series of $f(x) = \sin(x)$. We have

$$\begin{aligned}
\sin(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\
&= \sin(0) + \frac{\cos(0)}{1!}x + \frac{-\sin(0)}{2!}x^2 + \frac{-\cos(0)}{3!}x^3 + \frac{\sin(0)}{4!}x^4 + \cdots \\
&= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}
\end{aligned}$$

It's not hard to find the interval of convergence is $(-\infty, +\infty)$.

Example 30. We want to find the Maclaurin series of $f(x) = \cos(x)$. We have

$$\begin{aligned}
\sin(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\
&= \cos(0) + \frac{-\sin(0)}{1!}x + \frac{-\cos(0)}{2!}x^2 + \frac{-\sin(0)}{3!}x^3 + \frac{\cos(0)}{4!}x^4 + \cdots \\
&= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}
\end{aligned}$$

It's not hard to find the interval of convergence is $(-\infty, +\infty)$.

Example 31. We want to prove binomial theorem by expanding $(1 + x)^\alpha$. We have

$$\begin{aligned}
(1+x)^\alpha &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots \\
&= 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots \\
&= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n
\end{aligned}$$

where $\binom{\alpha}{n} = \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} & n = 1, 2, 3, \ldots \\ 1 & n = 0 \end{cases}$. By ratio test, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{\alpha - n}{n+1} x \right| = |x| < 1$$

Therefore the radius of convergence is $R = 1$ when $\alpha < 0$.

ii. **Taylor Theorem**

Having Taylor series of $f(x)$ at c, we are interested whether the series converges to $f(x)$ and whether the *degree n Taylor polynomial of $f(x)$ about c* is a good approximation of $f(x)$. We need Taylor theorem.

(Taylor Theorem) Let $f(x)$ be such that $f(x)$, $f'(x)$, $f''(x), \cdots, f^{(n+1)(x)}$ exists and continuous on an interval containing c, then for any x in that interval

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)(c)}}{n!}(x-c)^n + E_n(x)$$

where

$$E_n(x) = \begin{cases} \dfrac{f^{(n+1)(s)}}{(n+1)!}(x-c)^{n+1} & \text{(Language)} \\ \dfrac{1}{n!}\displaystyle\int_c^x f^{(n+1)}(t)(x-t)^n\, dt & \text{(Integral)} \\ O((x-c)^n) & \text{(Peano)} \end{cases}$$

is the remainder (error term).

Example 32. We want to find the remainder (error) $E_n(x)$ of the degree n Taylor polynomial $T_n(x)$ of $f(x) = e^x$ about 0. By Taylor theorem, we have

$$|E_n(x)| = \frac{e^s |x|^{n+1}}{(n+1)!}$$

where s is between 0 and x. Since

$$\lim_{n\to\infty} \frac{e^s |x|^{n+1}}{(n+1)!} = e^s \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

we know the Maclaurin series converges to e^x.

Problem Solving IV

Determine the convergence of a given series; Find the radius of convergence and interval of convergence of a given power series; Find the power series of a given function; Find the function of the given power series; Application: approximation and differential equation

Problem 9: Determine the convergence of a given series

Basically we are comparing (i) whether the terms in the series converge to 0 and (ii) how fast they converge to 0. A comparison between different functions for large n maybe helpful:

$$n^n > n! > a^n > n^a > \ln(n) > \sin(n)$$

Notice $|\sin(n)| \leq 1$ and $|\cos(n)| \leq 1$ and then we could treat them as constant when n is large.

- **Polynomial** $p(n)$ - $\dfrac{1}{n^p}$

  1. Determine whether the following series converges (If it is an alternating series, determine whether it absolutely converges or conditionally converges).

     (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+4n}{\sqrt{n^4+5}}$

     (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$

     (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

     (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^5}$

     (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+4)}{n^2}$

  2. Determine whether the following series converges (If it is an alternating series, determine whether it absolutely converges or conditionally converges).

     (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+4)}{n^2}$

     (b) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{n^2}-\frac{1}{n}\right)$

     (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt[n]{e}}{n}$

     (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+7}}$

(e) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\ln(\ln(n))}$

- **Geometric** - ar^n

3. Determine whether the following series converges (If it is an alternating series, determine whether it absolutely converges or conditionally converges).

   (a) $\sum_{n=1}^{\infty} \sqrt[n]{1,1}$

   (b) $\sum_{n=1}^{\infty} \frac{1+3^n}{4^n}$

   (c) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{4^n}$

   (d) $\sum_{n=1}^{\infty} \frac{6}{n9^n}$

4. Determine whether the following series converges (If it is an alternating series, determine whether it absolutely converges or conditionally converges).

   (a) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r+1}{r}\right)$ (Hint: you need discuss the value of r).

   (b) $\sum_{n=1}^{\infty} \frac{6^n+2^n}{7^n}$

   (c) $\sum_{n=1}^{\infty} \frac{2}{2^n+5}$

   (d) $\sum_{n=1}^{\infty} \frac{4^n}{2^n+3^n}$

5. Determine whether the following series converges (If it is an alternating series, determine whether it absolutely converges or conditionally converges).

   (a) $\sum_{n=1}^{\infty} \frac{(n+4)5^n}{n^2 4^{2n}}$

   (b) $\sum_{n=1}^{\infty} \frac{\sin(6n)}{6^n}$

   (c) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{5^n}$

   (d) $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2^n}\right)$

- n^n, $n!$

6. Determine whether the following series converges (If it is an alternating series, determine whether it absolutely converges or conditionally converges).
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!+1}{(n+1)!}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!3^n}{(4n)!}$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)^n}{7^{n^2}}$

7. Determine whether the following series converges (If it is an alternating series, determine whether it absolutely converges or conditionally converges).
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(4n)!}{n^{40}}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n}\right)^n$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)^n}{7^n}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n!}$

- **Special Series by Construction**

8. Determine whether the following converges.
   (a) Let $a_n = \begin{cases} \dfrac{1}{n} & \text{if } n \text{ does not contain the digit 7} \\ 0 & \text{otherwise} \end{cases}$. Consider $\displaystyle\sum_{n=1}^{\infty} a_n$.
   (b) Let $a_n = \begin{cases} \dfrac{1}{n} & \text{if } n \text{ does not contain the digit 0} \\ 0 & \text{otherwise} \end{cases}$. Consider $\displaystyle\sum_{n=1}^{\infty} a_n$.

Sometimes it is hard to determine how fast the terms converges to zero when the function is not that common. Then integral test would be our most powerful tool.

- **Integral Test**

8. Determine whether the following series converge by integral test.

   (a) $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n\ln(n)}$

   (b) $\displaystyle\sum_{n=6}^{\infty} 14ne^{-n^2}$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^3}$

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n}$

   (e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n+4}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$

9. Find the values of p for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{2p}}$ converges.

Problem 10: Find the radius of convergence and interval of convergence of a given power series

1. Find the interval of convergence of the following series

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-x)^n}{n^4 4^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+3^n)x^n}{n!}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^n}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{3n}}{n\ln(n)}$

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!!}$

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x-1)^n}{2^{2n+1}}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n x^n}{\ln(n+1)}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(2-x)^n}{n4^{2n}}$

2. Find the interval of convergence of the following series

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)3^n(x-6)^n}{6^{n+1}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!(-1)^n(x-3)^n}{(n-1)!9^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n n^2}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n(x+6)^{2n}}{9^{n+1}}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

   (h) $\displaystyle\sum_{n=0}^{\infty} \frac{n!(x-1)^n}{2^n}$

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(1-x)^n}{n!}$

   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-9)^n}{n(-5)^n}$

3. Let $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$ has a radius of convergence $R > 0$. Give an example that $\sum_{n=0}^{\infty} a_n$ converges and an example that $\displaystyle\sum_{n=0}^{\infty} a_n$ diverges.

4. Write down the power series with the interval of convergences.

(a) $[2, 8]$
(b) $(2, 8]$
(c) $[2, 8)$
(d) $(2, 8)$

Problem 11: Given a function, find the power series

There are two approaches to the answer. One way is to apply geometric series if $f(x)$ is a polynomial.

$$\begin{tikzcd}
\displaystyle\sum_{n=0}^{\infty} ar^n \arrow[r, leftrightarrow] \arrow[d, equal] & \displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n \arrow[d, equal] \\
\dfrac{a}{1-r} \arrow[r, leftrightarrow, "a=?", "r=?"'] & f(x)
\end{tikzcd}$$

Another way is to apply the Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

With these two main approaches, we also could apply some tricks/operations on the series: integration or differentiation term-by-term. composition, substitution and multiplication by x^n.

1. Find the power series with interval of convergence of the following functions (the center of convergence should be 0).
   (a) $f(x) = \dfrac{1}{1-x}$
   (b) $f(x) = \dfrac{1}{2-x}$
   (c) $f(x) = \dfrac{1}{1+x^2}$
   (d) $f(x) = \dfrac{3}{1-2x^2}$
   (e) $f(x) = \dfrac{x^2}{1+3x}$
   (f) $f(x) = \dfrac{x}{x^2+1}$
   (g) $f(x) = \dfrac{1+x}{1-x}$
   (h) $f(x) = \dfrac{2x+3}{x^2+3x+2}$
   (i) $f(x) = \dfrac{x^3}{4x^2+3}$
   (j) $f(x) = \dfrac{x+2}{2x^2-x-1}$

2. Find the power series with interval of convergence of the following functions (the center of convergence should be 0).

(a) $f(x) = \dfrac{1}{(1-x)^2}$
(b) $f(x) = \dfrac{1}{(1-x)^3}$
(c) $f(x) = \dfrac{1}{(1+x)^3}$
(d) $f(x) = \ln(1+x)$
(e) $f(x) = \arctan(x)$
(f) $f(x) = \ln(3+x)$
(g) $f(x) = \arctan(3x)$
(h) $f(x) = \dfrac{2x}{(1+x^2)^2}$

3. Find the power series with interval of convergence of the following functions (the center of convergence should be 0).
   (a) $f(x) = e^x$
   (b) $f(x) = \sin(x)$
   (c) $f(x) = \cos(x)$
   (d) $f(x) = (1+x)^\alpha$

4. Expand the following functions as power series as required and find the interval of convergence.
   (a) $f(x) = \ln(3+x)$ about $x = -2$
   (b) $f(x) = x\ln(x)$ about $x = 1$
   (c) $f(x) = \sin(x)$ about $x = \dfrac{\pi}{2}$
   (d) $f(x) = e^{3x}$ about $x = 1$
   (e) $f(x) = \sqrt{x}$ about $x = 25$

Problem 12: Given a function, find the power series

In practice, if we meet $p(n)$, try $\dfrac{1}{1-x}$; if we meet $\dfrac{1}{p(n)}$, try $\ln(1-x)$ or $\arctan(x)$; if we meet factorials $\dfrac{1}{n!}$, try e^x, $\sin(x)$ and $\cos(x)$. Usually the tricks of decomposition, differentiation and integration are applied.

1. Find the what functions the following power series converge to.
   (a) $\displaystyle\sum_{n=1}^{\infty} nx^{n-1}$
   (b) $\displaystyle\sum_{n=0}^{\infty} nx^{n}$

(c) $\displaystyle\sum_{n=0}^{\infty} n^2 x^n$

(d) $\displaystyle\sum_{n=0}^{\infty}(n+1)(n+4)x^n$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n n(n+1)}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1}$

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+1}{n!} x^n$

(h) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!(n+2)}$

2. Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!(n+2)}$.
3. Find $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

Problem 13: Application: approximation and differential equation

- **Approximation**
  1. Make a Taylor approximation with 3 non-zero terms of $e^{0.1}$ by e^x about $x=0$, e^{x+1} about $x=-1$ and about $x=-1$.
  2. Make a Taylor approximation with 4 non-zero terms of $\ln(0.9)$ by $\ln(x)$ about $x=1$, $\ln(x+1)$ about $x=0$ and $\ln(x-1)$ about $x=2$.
  3. Make a Taylor approximation with 3 non-zero terms of $\sqrt{24}$ by $\sqrt{x}$ about $x=25$, $\sqrt{x+1}$ about $x=24$ and $\sqrt{x-10}$ about $x=26$.
  4. Make a linear approximation of $\sqrt{2}$ by $\sqrt{x}$ about $x=4$ and about $x=1$. Compare which one is more accurate.
  5. Suppose $f(1)=2$ and $f'(x)=e^{x^2}$. Make a linear approximation of $f(1.1)$ and judge the approximation.
  6. Suppose the radius of a sphere is measured to be 10cm, with a maximum error of 0.001cm. Use an appropriate linear approximation to estimate the maximum error in the calculated volume of the sphere.

7. Integrate $\int_{-x}^{x} e^{-t^2}\,dt$ in terms of power series.
8. Integrate $\int_{0}^{x} \arctan(x^3)\,dx$ in terms of power series.
9. Integrate $\int_{0}^{\frac{\pi}{2}} 4a\sqrt{1-e^2\sin^2\theta}\,d\theta$ in terms of power series.
10. Evaluate the following limits.
    (a) $\lim_{x\to\infty} \frac{-x+\sin(x)}{x^4}$
    (b) $\lim_{x\to\infty} \frac{x^2-2+2\cos(x)}{x^4}$
    (c) $\lim_{x\to\infty} \frac{(e^{2x}-1)\ln(1+x^2)}{(1-\cos(3x))^2}$
11. Determine what degree n of Maclaurin polynomial $T_n(x)$, of the function $f(x) = \ln(1+x)$ is needed to guarantee that the Maclaurin polynomial approximation of $\ln(1.4)$ is accurate with in 10^{-3}.
12. Determine whether the following argument is true: Given the same centers, a higher degree Taylor polynomial is a more accurate approximation to a function than a lower degree approximation.

- **Differential Equation**

14. Let $f(x) = 2\sin(x)\cos(x)$, find $f^{(101)}(0)$.
15. Determine whether the following argument is true: A function which has a Taylor series representation is infinitely differentiable.
16. Find the solution of $f''(x) = -f(x)$.
17. Find the solution of $(1+x)f'(x) = \alpha f(x)$.