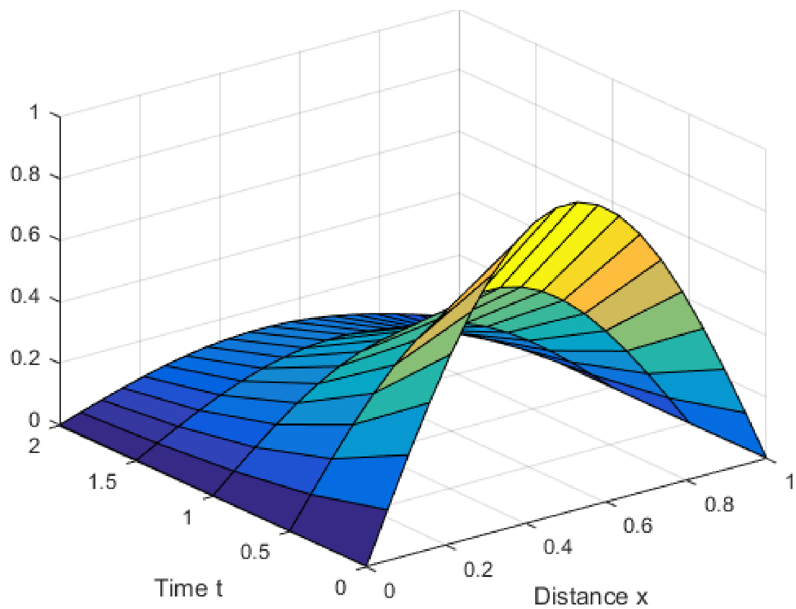


# MATH 316 Elementary Differential Equation II

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[1]The cover image is from https://www.mathworks.com/help/matlab/math/partial-differential-equations.html

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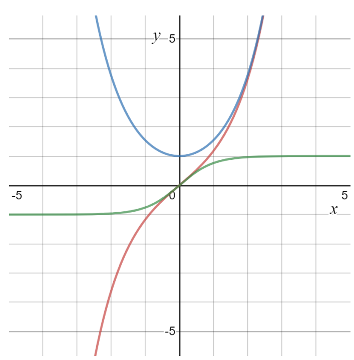
# Review of Techniques for ODEs

On the purpose of developing some new techniques in this course, we are going to review two related types of differential equations specifically.

1. **Second Order Linear ODE with Constant Coefficient**
Generally, the homogeneous second order linear ODE with constant coefficient has the form
$$ay'' + by' + cy = 0$$
which could be interpreted as $L[y] = 0$ where
$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$$
We guess the solution to be
$$y = e^{rx}$$
Plug into the
$$(ar^2 + br + c)e^{rx} = 0$$
which implies
$$ar^2 + br + c = 0$$
To solve it, we have three cases ($\Delta = b^2 - 4ac$):
   - $\Delta > 0$. We have two real roots $r_1$, $r_2$. And the solution would be
$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$
If $r_{1,2} = \pm r$, we could have
$$y(x) = C_1 \sinh(rx) + C_2 \cosh(rx)$$

**Hyperbolic function:** We define $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\tanh x = \dfrac{\sinh x}{\cosh x}$.



They have the following properties:

- $\cosh^2 x - \sinh^2 x = 1$
- $\cosh' x = \sinh x$
- $\sinh' x = \cosh x$
- $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
- $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

- $\Delta = 0$. We have a repeated real roots $r_0$. Then we know $y_1 = e^{r_0 x}$. To find the second solution, we plug it into the equation, we have $L\left[e^{rx}\big|_{r=r_0}\right] = 0$. Differentiate it, we have
$$0 = \frac{\partial}{\partial r} L\left[e^{rx}\right]\bigg|_{r=r_0} = L\left[\frac{\partial}{\partial r} e^{rx}\bigg|_{r=r_0}\right]$$
Therefore, we have the second solution to be
$$y_2 = \frac{\partial}{\partial r} e^{rx}\bigg|_{r=r_0} = x e^{r_0 x}$$
Therefore the solution would be
$$y(x) = C_1 e^{r_0 x} + C_2 x e^{r_0 x}$$
- $\Delta < 0$. We have two complex roots $\alpha \pm \beta i$. Then the solution would be
$$\begin{aligned} y(x) &= A e^{(\alpha+\beta i)t} + B e^{(\alpha-\beta i)t} \\ &= A e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) + B e^{\alpha t}(\cos(\beta t) - i\sin(\beta t)) \end{aligned}$$
Let $A = C' + iC''$, $B = C' - iC''$. Then let $C_1 = 2C'$, $C_2 = 2C''$. Then we have the solution to be
$$y(x) = e^{\alpha t}(C_1 \sin(\beta t) + C_2 \cos(\beta t))$$

2. **Cauchy-Euler Equation**
Generally, the Cauchy-Euler Equation is
$$x^2 y'' + \alpha x y' + \beta y = 0$$
which could be interpreted as $L[y] = 0$ where
$$L = x^2 \frac{d^2}{dx^2} + \alpha x \frac{d}{dx} + \beta$$
We guess the solution to be
$$y = x^r$$

Plug into the equation, we have

$$(r(r-1)+\alpha r+\beta)x^r=0$$

which is

$$r^2+(\alpha-1)r+\beta=0$$

To solve it, we have 3 cases ($\Delta=(\alpha-1)^2-4\beta$):

- $\Delta>0$. We have two real roots $r_1$, $r_2$. Then the solution would be

$$y(x)=C_1x^{r_1}+C_2x^{r_2}$$

- $\Delta=0$. We have a repeated root $r_0$. Then the first solution would be $y(x)=x^{r_0}$. To find the second solution, we plug it into the equation, we have $L\left[e^{rx}|_{r=r_0}\right]=0$. Differentiate it, we have

$$0=\frac{\partial}{\partial r}L\left[e^{rx}\right]\bigg|_{r=r_0}=L\left[\frac{\partial}{\partial r}e^{rx}\bigg|_{r=r_0}\right]$$

  Therefore, we have the second solution to be

$$y_2=\frac{\partial}{\partial r}x^r\bigg|_{r=r_0}=\ln|x|x^{r_0}$$

  Therefore the solution would be

$$y(x)=C_1x^{r_0}+C_2\ln|x|x^{r_0}$$

- $\Delta<0$. We have two complex roots $\lambda\pm i\mu$. Then the solution would be

$$\begin{aligned}y(x)&=Ax^{\lambda+i\mu}+Bx^{\lambda-i\mu}\\&=Ax^\lambda(\cos(\mu\ln|x|)+i\sin(\mu\ln|x|))+Bx^\lambda(\cos(\mu\ln|x|)-i\sin(\mu\ln|x|))\\&=(A+B)x^\lambda\cos(\mu\ln|x|)+(A-B)i\sin(\mu\ln|x|)\end{aligned}$$

  With $A+B=C_1$ and $(A-B)i=C_2$, we have

$$y(x)=C_1x^\lambda\cos(\mu\ln|x|)+C_2x^\lambda\sin(\mu\ln|x|)$$

## Series Solution to ODE's

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1. **Ordinary point and Singular Point**
   Consider the following linear second order differential equation with variable coefficients
$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad (*)$$
   If $P(x)$ has a solution $x_0$ such that $P(x_0) = 0$, in the neighborhood of $x_0$, we have to take the risk that there is no series solution around $x_0$. This could be observed via the equivalent form $y'' + \dfrac{Q(x)}{P(x)}y' + \dfrac{R(x)}{P(x)}y = 0$ where $\dfrac{Q(x)}{P(x)}$ and $\dfrac{R(x)}{P(x)}$ are possible to be not analytic (can not expanded around $x_0$).

   Based on this, we make the following definition. If $P(x_0) = 0$, we call $x_0$ to be *singular point*; If $P(x_0) \neq 0$, we call $x_0$ to be *ordinary point*.

   However, it is still possible that we can find a series solution around $x_0$. Inspired the idea of Cauchy-Euler equation, we divide the equation by $P(x)$ and multiplies $(x - x_0)^2$. We could get
$$(x - x_0)^2 y'' + (x - x_0)\left(\frac{Q(x)}{P(x)}(x - x_0)\right)y' + \left(\frac{R(x)}{P(x)}(x - x_0)^2\right)y = 0$$
   Let $p(x) = \dfrac{Q(x)}{P(x)}(x - x_0)$ and $q(x) = \dfrac{R(x)}{P(x)}(x - x_0)^2$. Then the equation becomes to be
$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0 \qquad (**)$$
   If $p(x)$ and $q(x)$ are analytic at $x = x_0$, we call $x_0$ *regular singular point* which is possible to have a series solution around $x = x_0$; if $p(x)$ or $q(x)$ are not analytic at $x = x_0$, we call $x_0$ *irregular singular point*.

2. **Series Solution near Ordinary Points**
   First of all, let's look at an example of ordinary points. We have to find the **recurrence relationship** to find the **series** $a_n$ and **radius of convergence**.

   **Example 1.** Find the series solution of $(x - 1)y'' + y' = 0$ around $x = 0$.
   *Solution.* Assume $y = \sum_{n=0}^{\infty} a_n x^n$, implying $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n a_n (n-1) x^{n-2}$.
   Plug in to the equation and simplify it, we have
$$\sum_{n=2}^{\infty} n a_n (n-1) x^{n-1} - \sum_{n=2}^{\infty} n a_n (n-1) x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

Then we shift the index and simplify to get

$$a_1 - 2a_2 + \sum_{n=2}^{\infty}(n^2 a_n - (n^2+n)a_{n+1})x^{n-1} = 0$$

By linear independence, we have $a_1 - 2a_2 = 0$ and $na_n - (n+1)a_{n+1} = 0$ for $n \geq 1$. Therefore we know $a_n = \dfrac{1}{n}$ $(n \geq 1)$ and $a_0$ could be arbitrary. The series solution is

$$y(x) = a_0 + a_1 \sum_{n=0}^{\infty} \frac{x^n}{n}$$

With ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = |x| \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = |x| < 1$$

we know the radius of convergence is $\rho = 1$. Then we test the endpoints. If $x = 1$, the series is $a_0 + a_1 \sum_{n=0}^{\infty} \dfrac{1}{n}$ which is a harmonic series and diverges. If $x = -1$, the series is $a_0 + a_1 \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n}$ which is an alternative harmonic series and converges.

It is not hard to find that in this example $\rho = 1$ is also the distance between the ordinary expansion point $x = 0$ and the singular point $x = 1$. Actually the more general statement is that if the expansion point for the series $x_0$ is ordinary then the radius of convergence is $\rho \geq |x_{\text{sp}} - x_0|$ where $x_{\text{sp}}$ is the nearest singular point - restricted by the radii of convergence of $\dfrac{Q(x)}{P(x)}$ and $\dfrac{R(x)}{P(x)}$ which are obviously bounded by $P(x_0) = 0$

3. **Series Solution near Regular Singular Points**
Consider the equation $(**)$, we expand $p(x)$ and $q(x)$ around a regular singular point.

$$p(x) = p_0 + p_1(x - x_0) + p_2(x - x_0)^2 + \cdots$$
$$q(x) = q_0 + q_1(x - x_0) + q_2(x - x_0)^2 + \cdots$$

It is obvious that $\lim_{x\to x_0} p(x) = p_0$ and $\lim_{x\to x_0} q(x) = q_0$. Since we are trying to find the series solution near $x_0$, the equation (**) could be approximated as a Cauchy-Euler equation

$$(x - x_0)^2 y'' + (x - x_0)p_0 y' + q_0 y = 0$$

with the solutions in the form $y(x, r) = (x - x_0)^r$. Therefore, the complete form could be written as

$$y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$$

which is known as a *Frobenius Series*. However, if $r$ is repeated or gaped by an integer, we can only find one solution directly, we have to use $\left.\dfrac{d}{dr} y(x, r)\right|_{r=r_0}$ to find the second solution.

Let's look at an example.

**Example 2.** Find the series solution of $2x^2y'' - xy' + (1-x)y = 0$ around $x_0 = 0$.

*Solution.* Although $P(0) = 0$, we could find $p(x) = \dfrac{Q(x)}{P(x)}(x-0) = -\dfrac{1}{2}$ and $q(x) = \dfrac{R(x)}{P(x)}(x-0)^2 = \dfrac{1-x}{2}$ are analytic then $\lim\limits_{x\to x_0} p(x) = -\dfrac{1}{2}$ and $\lim\limits_{x\to x_0} q(x) = \dfrac{1}{2}$. Then the corresponding Cauchy-Euler equation is $x^2y'' - \dfrac{x}{2} + \dfrac{1}{2}y = 0$ whose indicial equation is $r^2 - \dfrac{3}{2}r + \dfrac{1}{2} = 0$.

Assuming $y = \sum\limits_{n=0}^{\infty} a_n x^{n+r}$ with $y = \sum\limits_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y = \sum\limits_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$. We plug them in,

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

Shift the index, we have

$$a_0[2r(r-1) - r + 1]x^r + \sum_{n=1}^{\infty}\left[2a_n(n+r)(n+r-1) - a_n(n+r) + a_n - a_{n-1}\right]x^{n+r} = 0$$

We have $2r(r-1) - r + 1 = 0$ which is identical to the indicial equation with the solution $r = 1, \dfrac{1}{2}$. We also have $a_n((n+r)(2(n+r)-3)+1) = a_{n-1}$ $(n \geq 0)$ and $a_0$ is arbitrary. Then we would apply the fact $1 \times 3 \times \cdots \times (2n-1) = \dfrac{1 \times 2 \times 3 \times \cdots \times (2n)}{2 \times 4 \times 6 \times \cdots \times (2n)} = \dfrac{(2n)!}{2^n n!}$. If $r = 1$, the recurrence equation becomes $a_n = \dfrac{a_{n-1}}{(n+1)(2n-1)+1} = \dfrac{a_{n-1}}{2n^2+n} = \dfrac{a_{n-1}}{n(2n+1)}$. Then $a_n = \dfrac{a_n}{n! \times (1 \times 3 \times \cdots \times (2n+1))} = \dfrac{2^n a_0}{(2n+1)!}$. If $r = \dfrac{1}{2}$, the recurrence equation becomes $a_n = \dfrac{a_{n-1}}{(2n-1)n}$. Then $a_n = \dfrac{a_0}{n! \times (1 \times 3 \times \cdots \times (2n-1))} = \dfrac{2^{n-1}a_0}{n(2n-1)!}$. Therefore the solution is

$$y(x) = C_1y_1(x) + C_2y_2(x) = C_1x^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{2^{n-1}a_0}{n(2n-1)!} + C_2x\sum_{n=0}^{\infty}\frac{2^n a_0}{(2n+1)!}$$

With ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x|\lim_{n\to\infty}\frac{((n+r)(2(n+r)-3)+1)}{((n+r+1)(2(n+r)-1)+1)} = 0 < 1$$

we know the raius of convergence is $\rho = \infty$

Let's look at another more complicated example.

**Example 3.** (Frobenius Series: Bessel's Equation)
Consider the Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{*}$$

around $x = 0$. Since it is already in the required form, we know $P(x_0) = x_0^2 = 0$, $p(x) = 1$ and $q(x) = x^2 - v^2$. Since $p(x)$ and $q(x)$ are analytic, we know $x_0 = 0$ is a regular singular point. Take the limit, we have

$$p_0 = \lim_{x\to 0} p(x) = 1,\ q_0 = \lim_{x\to 0} q(x) = -\nu^2$$

Then indicial equation is

$$r(r-1) + r - \nu^2 = r^2 - \nu^2 = 0$$

with the solution $r = \pm\nu$. Then we would try to let $y = \sum_{n=1}^{\infty} a_n x^{n+r}$ and plug in (*), we have

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

We want to shift the index to be consistent

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Simplify the equation, we have

$$\sum_{n=2}^{\infty}\left(a_n((n+r)^2 - \nu^2) + a_{n-2}\right)x^{n+r} + a_0(r^2 - \nu^2)x^r + a_1((r+1)^2 - \nu^2)x^{r+1} = 0$$

By linear independence, we have

$$\begin{cases} r^2 - \nu^2 = 0 & \text{if } a_0 \neq 0 \\ a_1((r+1)^2 - \nu^2) = 0 & \\ a_n((n+r)^2 - \nu^2) + a_{n-2} = 0 & \end{cases}$$

We have to discuss the root cases based whether two roots are the same or gaped by integers. Then $\mu$ could be classified as $\nu \in \mathbb{Z}$, $\nu = 0$, $\nu = \dfrac{m}{2}$ and $\nu \notin \mathbb{Z} \cap \nu \neq \dfrac{m}{2} \cap \nu \neq 0$.

- **Bessel Equation of Order Zero** The equation is reduced to

  $$x^2y'' + xy' + x^2y = 0$$

  and the roots to the indicial equation is $r_{1,2} = 0$. Then we know $a_0$ could be arbitrary and $a_1 = 0$. And the recursion relation is

  $$a_n = -\frac{a_{n-2}}{n^2}$$

so we know

$$a_{2k} = \frac{(-1)^k}{(2k)^2(2k-2)^2\cdots 2^2}a_0 = \frac{(-1)^k a_0}{2^{2k}(k!)^2}$$

and

$$a_{2k+1} = 0$$

Then our first solution is

$$y_1(x) = \sum_{k=1} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} = J_0(x)$$

With the derivative in $r$ of

$$y(x,r) = a_0 x^r \left(1 - \frac{x^2}{(2+r)^2} + \cdots + \frac{(-1)^k x^{2k}}{(2+r)^2\cdots(2k+r)^2} + \cdots\right)$$

we find the second the solution would be

$$\left.\frac{\partial y}{\partial r}\right|_{r=r_1} = a_0 \ln x y_1(x) + a_0 x^r \sum_{k=1}^{\infty} x^{2k} \frac{\partial}{\partial r}\left(\frac{1}{(2+r)^2\cdots(2k+r)^2}\right)$$

Let $f_k(r) = \dfrac{1}{(2+r)^2\cdots(2k+r)^2}$. Since

$$\ln f_k(r) = -2\left(\ln(2+r) + \cdots + \ln(2k+r)\right)$$

we have

$$\frac{f_k'(0)}{f_k(0)} = \frac{d}{dr}\ln f_k(0) = -\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) = H_k$$

Then we have

$$y_2(x) = J_0(x)\ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} H_k}{2^{2k}(k!)^2} x^{2k}$$

In convention, it is defined as

$$Y_0(x) = \frac{2}{\pi}\left(y_2(x) + (\gamma - \log 2) J_0(x)\right)$$

where $\gamma$ is the Euler constant. Then the complete solution is

$$y(x) = c_1 J_0(x) + c_2 Y_0(x)$$

- **Bessel Equation of Order Multiple-Half** For simplicity we let $\nu = \dfrac{1}{2}$. The equation is reduced to

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

and the roots to the indicial equation is $r_{1,2} = \pm\dfrac{1}{2}$. For $r = \dfrac{1}{2}$ we know $a_0$ could be arbitrary and $a_1 = 0$. And the recursion relation is

$$a_n = -\frac{a_{n-2}}{\left(n + \frac{1}{2}\right)^2 - \frac{1}{4}} = -\frac{a_{n-2}}{(n+1)n}$$

so we know

$$a_{2k} = \frac{(-1)^k}{(2k+1)2k\cdots 1} a_0 = \frac{(-1)^k a_0}{(2k+1)!}$$

and

$$a_{2k+1} = 0$$

Then the solution is

$$y_1(x) = x^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} = x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x^{-\frac{1}{2}} \sin x$$

If $r = -\dfrac{1}{2}$, we know $a_0$ and $a_1$ are arbitrary. And the the recursion equation is

$$a_n = -\frac{a_{n-2}}{\left(n - \frac{1}{2}\right)^2 - \frac{1}{4}} = -\frac{a_{n-2}}{n(n-1)}$$

Then we know

$$a_{2k} = \frac{(-1)^k a_0}{2k(2k-1)\cdots 1} = \frac{(-1)^k a_0}{(2k)!}$$

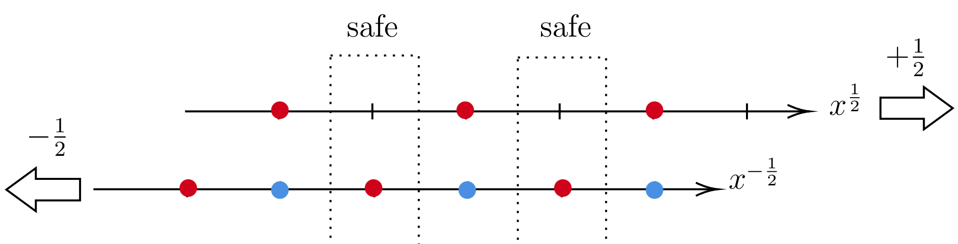
and

$$a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)2k\cdots 1} = \frac{(-1)^k a_1}{(2k+1)!}$$

Then the second solution is

$$y_2(x) = a_0 x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + a_1 x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = a_0 x^{-\frac{1}{2}} \cos x + a_1 x^{-\frac{1}{2}} \sin x$$

We can see $y_1(x)$ is included in $y_2(x)$. In this part we find that we don't have to use the derivative by $r$. This is because the gap between two terms are 2 but the gap between two roots is one. With visualization, it is




- **Bessel Equation of Order Integer** For simplicity, we choose $\nu = 1$. The equation is reduced to

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

and the roots to the indicial equation is $r_{1,2} = \pm 1$. Then we know $a_0$ could be arbitrary and $a_1 = 0$. And the recursion relation is

$$a_n = -\frac{a_{n-2}}{n(n+2)}$$

Then we know

$$a_{2k} = \frac{(-1)^k a_0}{(2k+2)(2k)\cdots 1(2k)(2k-2)\cdots 1} = \frac{(-1)^n a_0}{2^{2k}(k+1)!k!}$$

Then, with letting $a_0 = \frac{1}{2}$, the first solution is

$$y_1(x) = \frac{x}{2}\sum_{k=0}^{\infty} \frac{(-1)^n a_0 x^{2k}}{2^{2k}(k+1)!k!} = J_1(x)$$

To find the second solution, we play the same trick, getting

$$y_2(x) = aJ_1(x)\ln x + x^{-1}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right)$$

Then we plug $y_2(x)$ back to the equation,

$$-c_1 + c_0 x + \sum_{n=2}^{\infty}((n^2-1)c_{n+1} + c_{n-1})x^n = -a\left(x + \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)x^{2n+1}}{x^{2n}(n+1)!n!}\right)$$

where $c_0 = 1$. Then we choose $c_2 = \frac{1}{2^2}$ and we find

$$c_{2n} = \frac{(-1)^{n+1}(H_n + H_{n-1})}{2^{2n}n!(n-1)!}$$

Therefore the second solution is

$$y_2(x) = -J_1(x)\ln x + \frac{1}{x}\left(1 - \sum_{n=1}^{\infty} \frac{(-1)^n(H_n + H_{n-1})x^{2n}}{2^{2n}n!(n-1)!}\right)$$

In convention we we define

$$Y_1(x) = \frac{2}{\pi}\left(-y_2(x) + (\gamma - \ln 2)J_1(x)\right)$$

And the complete solution is

$$y = c_1 J_1(x) + c_2 Y_1(x)$$

- **Bessel Equation of Other Orders** In general the equation is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

and the roots to the indicial equation is $r_{1,2} = \pm\nu$. If $r = \nu$ we know $a_0$ could be arbitrary and $a_1 = 0$. And the recursion relation is

$$a_n = -\frac{a_{n-2}}{n(n+2\nu)}$$

Then we know

$$a_{2k} = \frac{(-1)^k a_0}{k!2^{2k}(1+\nu)\cdots(n+\nu)}$$

and

$$a_{2k+1} = 0$$

Then the first solution is

$$y_1(x) = x^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k a_0 x^{2k}}{k!2^{2k}(1+\nu)\cdots(n+\nu)}$$

If $r = -\nu$, then know $a_0$ could be arbitrary and $a_1 = 0$. And the recursion relation is

$$a_n = -\frac{a_{n-2}}{n(n-2\nu)}$$

Then we know

$$a_{2k} = \frac{(-1)^k a_0}{k!2^{2k}(1-\nu)\cdots(n-\nu)}$$

and

$$a_{2k+1} = 0$$

Then the second solution is

$$y_2(x) = x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k a_0 x^{2k}}{k!2^{2k}(1-\nu)\cdots(n-\nu)}$$

The complete solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

**Summary:** In summary, if we have two roots $r_1 \geq r_2$ and the first solution is

$$y_1(x) = \sum_{n=1}^{\infty} a_n (x-x_0)^{n+r_1}$$

then the second solution is

- <u>CASE 1</u>: If $r_1 - r_2$ is neither 0 or positive integer,

  $$y_2(x) = \sum_{n=1}^{\infty} b_n (x-x_0)^{n+r_2}$$

  where $b_0 = 1$.
- <u>CASE 2</u>: If $r_1 - r_2 = 0$,

  $$y_2(x) = y_1(x)\ln(x-x_0) + \sum_{n=1}^{\infty} b_n (x-x_0)^{n+r_2}$$

  for some $b_1, b_2, \ldots$.

- <u>CASE 3</u>: If $r_1 - r_2$ is a positive integer,

$$y_2(x) = a y_1(x) \ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$$

where $b_0 = 1$.

# Introduction to Partial Differential Equation

1. **Definition and Classification**
   Besides ordinary differential equation, we could have *partial differential equation* Involving multivariable functions $u(x,t)$,$u(x,y)$ that are determined by prescribing a relationship between the function value and its partial derivatives. For examples, with $u(x,y)$ as a function, we have first order linear PDE,

   $$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y)$$

   first order non-linear PDE,

   $$a(x,y,u)u_x + b(x,y,u)u_y + c(x,y,u)u = d(x,y,u)$$

   and second order linear PDE

   $$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

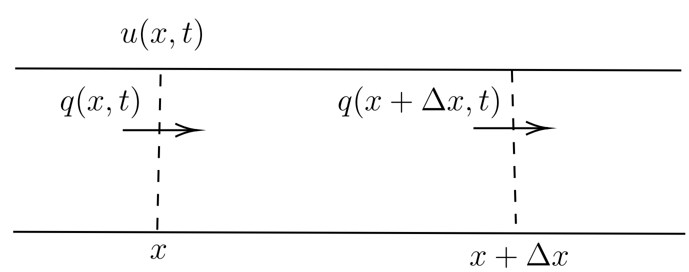
   where $A, B, C, D, E, F, G$ are all constants.

   This course is mainly focused on the second order linear PDE. Therefore we are going to do some classification before starting. The classification is inspired from the analogy of the quadratic surface $Ax^2 + Bxy + Cy^2 + Dx + Ey = K$ which could be classified by $\Delta = B^2 - 4AC$.

| $\Delta$ | Type | Quadratic | PDE | Nature |
|---|---|---|---|---|
| $\Delta = 0$ | Parabolic | $X^2 = T$ | $u_t = u_{xx}$ | Heat/Diffusion |
| $\Delta < 0$ | Elliptic | $X^2 + Y^2 = K$ | $u_{xx} + u_{yy} = f(x,y)$ | Poisson if $f \neq 0$<br>Laplace if $f = 0$ |
| $\Delta > 0$ | Hyperbolic | $T^2 = c^2X^2$ | $u_{tt} = c^2u_{xx}$ | wave |

   All linear second order PDE's can be transferred into one of these types.

2. **Models and Examples**
   We will consider the situations of transportation obeying ***conservation laws***.
   First let's look at a one dimensional case. Consider some matter flowing in one dimension. Let $u(x,t)$ to be the density of the matter and $q(x,t)$ to be the flux of the matter (we define flow from left to right is positive).



It could be approximated that the amount change of matter between $x$ and $x + \Delta x$ equals to $[u(x, t + \Delta t) - u(x, t)]\Delta$ and also equals to $[q(x, t) - q(x + \Delta x)]\Delta t$. Therefore,

$$[u(x, t + \Delta t) - u(x, t)]\Delta x = [q(x, t) - q(x + \Delta x)]\Delta t$$

Take the limit $\Delta x \to 0$ and $\Delta t \to 0$, we have $\dfrac{\partial u}{\partial t} = -\dfrac{\partial q}{\partial x}$. Rearrange it, we have

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$$

To describe the model better, we have to find the relationship between $u$ and $q$. Let's look at some examples.

**Example 4.** Convection and the first order Wave Equation
If we let $q = cu$, $c > 0$, we could get

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

We could guess a solution

$$u(x, t) = e^{ikx+\delta t}$$

Plug into the equation, $(\delta + cik)e^{ikx+\delta t}$, we have the dispersion equation

$$\delta = -ick$$

More generally, the solution is a right moving wave

$$u(x, t) = f(x - ct)$$

by plugging in $u_t + cu_x = -cf' + cf' = 0$. This wave could be observed in two reference frames: stationary observer $x$ and moving observer $x' = x - ct$ with speed $c$ (moving with the wave). Therefore, we have

$$f(x') = f(x - ct)$$

The linear relation could also have a negative coefficient, $q = -cu$, $c > 0$, which gives

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0$$

a left moving wave. If we combine these two operators $\left(\dfrac{\partial}{\partial t} + c\dfrac{\partial}{\partial x}\right)\left(\dfrac{\partial}{\partial t} - c\dfrac{\partial}{\partial x}\right)u = 0$, we could get the second order wave equation

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial^2 u}{\partial x^2} = 0$$

**Example 5.** Heat Conduction and Diffusion
Consider the diffusion of molecules, by *Fick's Law*

$$q = -\alpha^2 \frac{\partial u}{\partial x}$$

we would have

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Consider the heat conduction, by *Fourier's Law*

$$q = -k\frac{\partial T}{\partial x}$$

and the relationship between energy (heat) and temperature

$$u = \rho C T$$

we would also have

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

where $\alpha^2 = \dfrac{k}{\rho C}$.

We could make a guess here $u = e^{ikx+\delta t}$. Plug into the equation, $\delta e^{ikx+\delta t} = -k^2\alpha^2 e^{ikx+\delta t}$, we get the dispersion equation

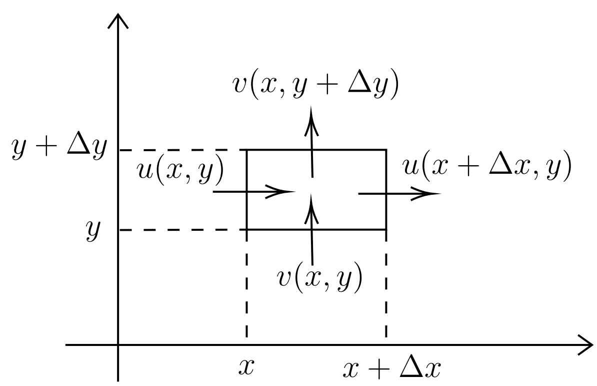
$$\delta = -k^2\alpha^2$$

The heat conduction/diffusion equation could also be in higher dimension

$$\frac{\partial u}{\partial t} = \triangle u$$

where $\triangle = \nabla^2$.

Then let's look at the two dimensional case. Since including $t$ would give three variables, we consider the steady state of flow. Let $u(x, y)$ be the $x$ component of the velocity and $v(x, y)$ to the $y$ component of the velocity. Let $\rho$ be the density of the material.




Since it is steady, $\rho$ should does not change. Therefore the mass change should be zero,

$$\rho[v(x, y + \Delta y) - v(x, y)]\Delta x + \rho[u(x + \Delta x, y) - u(x, y)]\Delta y = 0$$

Take the limit $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

**Example 6.** Laplace Equation
Consider the flow in the porous media, we have $h(x, y)$ to be the hydraulic head, then by *Darcy's law*

$$u = -k\frac{\partial h}{\partial x},\ v = -k\frac{\partial h}{\partial y}$$

where $k$ is hydraulic conductivity. Therefore, we have

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

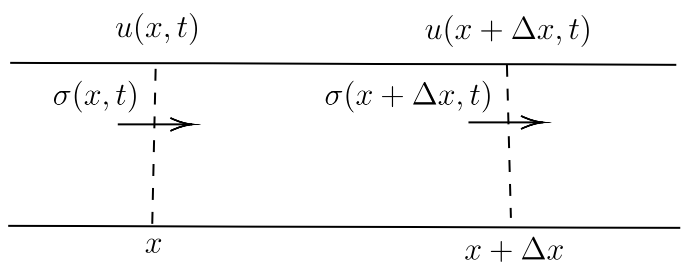
With the idea as the previous example, we could generalize the spacial differentiation,

$$\triangle h = 0$$

There is one thing to notice, Fick's Law, Fourier's Law and Darcy's law are physically the same law.

Besides the transportation, let's consider the wave equation.

**Example 7.** Wave Equation
Let $u(x, t)$ be the wave equation, $\sigma(x, t)$ be the stress in the rod and $\rho$ be the density of the rod.




By *Newton Second Law* $F = ma$, we have

$$\underbrace{[\sigma(x+\Delta x, t) - \sigma(x, t)]\Delta x}_{\text{force}} = \underbrace{\rho\Delta x}_{\text{mass}} \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}}$$

Take the limit $\Delta x \to 0$, we have

$$\frac{\partial \sigma}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}$$

By *Hook's Law* and *Young's Modulus*

$$\sigma = E\frac{\partial u}{\partial x},\ E > 0$$

we have the equation

$$\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial t^2},\ c = \sqrt{\frac{\rho}{E}}$$

With the idea as the two previous examples, we could generalize the spacial differentiation,

$$\triangle u = c^2\frac{\partial^2 u}{\partial t^2}$$

3. **Initial Conditions (For Time)**
   (a) First Order in Time
   $$u(x,0) = f(x)$$
   (b) Second Order in Time
   $$u(x,0) = f(x),\ u_t(x,0) = g(x)$$

   **Example 8.** Initial Conditions of Wave Equation
   Consider a one dimensional wave equation
   $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$
   Based on the second order derivative of time, we expect to have two initial conditions. Therefore we have
   $$u(x,0) = f(x),\, u_t(x,0) = g(x)$$

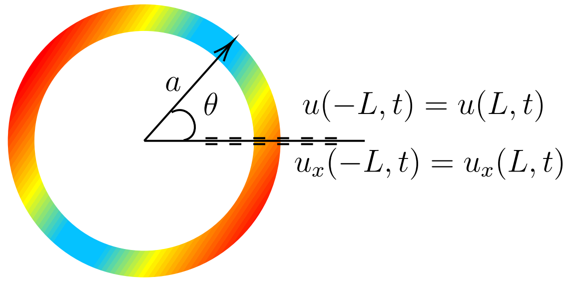
4. **Boundary Conditions (For Space)**
   We have $0 < x < L$.
   (a) Dirichlet
   $$u(0,t) = f(t),\, u(L,t) = g(t)$$
   (b) Neumann
   $$u_x(0,t) = f(t),\, u_x(L,t) = g(t)$$
   (c) Mixed
   $$u(0,t) = f(t),\, u_x(L,t) = g(t)$$
   (d) Periodic
   $$u(0,t) = u(L,t)$$

   **Example 9.** Mixed Boundary Condition of Heat Equation
   Consider the heat conduction
   $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
   in a rod with one end to be ice with constant temperature 0°C and one end to be isolated with no heat flow.

   Based on the second derivative of space, we expect to have two boundary conditions, therefore we have
   $$u_x(0,t) = u(L,t) = 0$$

**Example 10.** Periodic Boundary Condition of Heat Equation
Consider the heat conduction in the circular rod.




We know in the curvilinear coordinate (polar coordinate), the Laplace operation

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{a^2}\frac{\partial^2}{\partial \theta^2}$$

Then the heat equation would be

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{a^2}\frac{\partial^2 u}{\partial \theta^2}$$

However we assume there is only angular heat transform without radial heat transform $r \equiv a$, so we have $\dfrac{\partial u}{\partial r} = \dfrac{\partial^2 u}{\partial r^2} = 0$. Then we have

$$\frac{\partial u}{\partial t} = \frac{1}{a^2}\frac{\partial^2 u}{\partial \theta^2}$$

Let $x = a\theta$ to be the arc length, we have

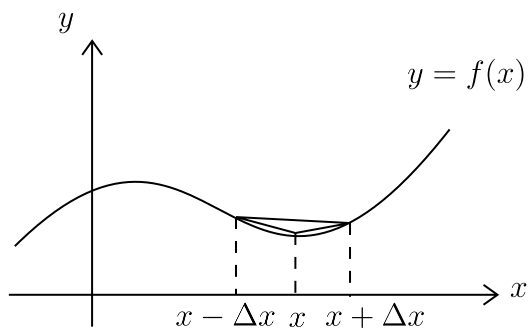
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Based on the second derivative of space, we expect to have two boundary conditions, therefore we have

$$u(-L,t) = u(L,t),\ u_x(-L,t) = u_x(L,t)$$

## Introduction to Numerical Method of PDE

1. **Finite Difference and Approximation of Derivatives**
Consider the finite difference $\Delta x$ away from $x$ on $f(x)$.




We could have some approximation by Taylor expansion,

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2}\Delta x^2 + \frac{f^{(3)}(x)}{6}\Delta x^3 + O(\Delta x^4) \tag{*}$$

$$f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{f''(x)}{2}\Delta x^2 - \frac{f^{(3)}(x)}{6}\Delta x^3 + O(\Delta x^4) \tag{**}$$

We could estimate the first order derivative in three different ways.

- Forward difference: Rearrange (*) we could pull out $f'(x)$,

$$\begin{aligned} f'(x) &= \frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{f''(x)}{2}\Delta x - \frac{f^{(3)}(x)}{6}\Delta x^2 + O(\Delta x^3) \\ &= \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x) \end{aligned}$$

  with the first order accuracy.

- Backward difference: Rearrange (**) we could pull out $f'(x)$,

$$\begin{aligned} f'(x) &= \frac{f(x) - f(x-\Delta x)}{\Delta x} - \frac{f''(x)}{2}\Delta x + \frac{f^{(3)}(x)}{6}\Delta x^2 + O(\Delta x^3) \\ &= \frac{f(x) - f(x-\Delta x)}{\Delta x} + O(\Delta x) \end{aligned}$$

  with the first order accuracy.

- Centered difference: With $(*) - (**)$ and rearrangement, we have

$$\begin{aligned} f'(x) &= \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - \frac{f^{(3)}(x)}{6}\Delta x^2 + O(\Delta x^3) \\ &= \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + O(\Delta x^2) \end{aligned}$$

  with the second order accuracy.

We can find that the centered difference is the most accurate,

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

Therefore we would estimate the second derivative with centered difference. We have $(*) + (**)$ and rearrange it,

$$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} + O(\Delta x^2)$$

with the second order accuracy.

2. **Heat Equation**
We know the partial differential equation describing heat transformation or diffusion is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L$$

Discretize it with forward difference because the physical phenomenon is going forward,

$$\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = \alpha^2 \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{\Delta x^2}$$

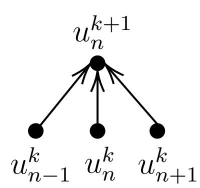
The error is first order in time and second order in space.
To simplify, we use the index representation

$$u_n^k = u(n\Delta x, k\Delta t)$$

where $0 \le n \le \frac{L}{\Delta x} = N$ and $0 \le k \le \frac{T}{\Delta t} = K$ ($0 < t < T$ is the time we want to solve).
Then the discretized equation could be rearranged as

$$u_n^{k+1} = u_n^k + \frac{\alpha^2 \Delta t}{\Delta x^2}(u_{n+1}^k - 2u_n^k + u_{n-1}^k) \tag{A}$$

which is the recursion relation.




To make this method to be stable, we need the coefficient, $\frac{\alpha^2 \Delta t}{\Delta x^2} \le \frac{1}{2}$.
Then what we have to do is apply the initial condition and boundary condition. The initial condition is usually $u(x,0) = f(x)$. With discretization, the initial condition is

$$u_n^0 = f(n\Delta x)$$

There are two types of boundary conditions. If it is Dirichlet, then we could have

$$u_0^{k+1} = u_0^k = A, u_N^{k+1} = u_N^k = B$$

If it is Neumann

$$\frac{\partial}{\partial x}u(0,t) = C$$

then we could have, with discretization,

$$\frac{u(\Delta x,t) - u(-\Delta x,t)}{2\Delta x} = C$$

which is $u_{-1}^k = u_1^k - 2\Delta x C$ in index form. Plug it into $(A)$, we have

$$u_1^{k+1} = u_1^k + \frac{\alpha^2 \Delta t}{\Delta x^2}(2u_1^k - 2u_0^{k+1} - 2\Delta x C)$$

Similarly at $x = L$ we have $u_{N+1}^k = u_{N-1}^k + 2\Delta x C$ and plugging into $(A)$ gives

$$u_N^{k+1} = u_N^k + \frac{\alpha^2 \Delta t}{\Delta x^2}(2u_{N-1}^k - 2u_N^{k+1} + 2\Delta x C)$$

In summary, the relationship could be written in the matrix-vector product form. If the boundary condition is Dirichlet, we have $\overrightarrow{v}_{k+1} = A\overrightarrow{v}_k$

$$\begin{pmatrix} u_0^{k+1} \\ u_1^{k+1} \\ \vdots \\ u_{N-1}^{k+1} \\ u_N^{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & & & 0 \\ r & -2r & r & \cdots & & & 0 \\ & & & \ddots & \ddots & & \\ & & & & r & -2r & r \\ & & & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0^k \\ u_1^k \\ \vdots \\ u_{N-1}^k \\ u_N^k \end{pmatrix}$$

where $r = \dfrac{\alpha^2 \Delta t}{\Delta x^2}$. If the boundary condition is Neumann, we have $\overrightarrow{v}_{k+1} = A\overrightarrow{v}_k + \overrightarrow{b}$

$$\begin{pmatrix} u_0^{k+1} \\ u_1^{k+1} \\ \vdots \\ u_{N-1}^{k+1} \\ u_N^{k+1} \end{pmatrix} = \begin{pmatrix} -2r & 2r & 0 & \cdots & & & 0 \\ r & -2r & r & \cdots & & & 0 \\ & & & \ddots & \ddots & & \\ & & & & r & -2r & r \\ & & & & 0 & 2r & -2r \end{pmatrix} \begin{pmatrix} u_0^k \\ u_1^k \\ \vdots \\ u_{N-1}^k \\ u_N^k \end{pmatrix} + \begin{pmatrix} -2\Delta x C r \\ 0 \\ \vdots \\ 0 \\ 2\Delta x C r \end{pmatrix}$$

To solve the equation, we need to iterate $k$ from 0 to $K$ while applying the recursion equation on initial condition repeatedly.

3. **Wave Equation**

We know the partial differential equation describing wave is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L$$

With discretization by centered difference, we have

$$\frac{u(x,t+\Delta t) - 2u(x,t) + u(x,t-\Delta t)}{\Delta t^2} = c^2 \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2}$$

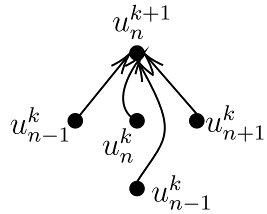
The error is second order in time and second order in space.
With index representation, we have

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\Delta t^2} = c^2 \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2}$$

Solve for $u_n^{k+1}$, we get

$$u_n^{k+1} = \frac{c^2 \Delta t^2}{\Delta x^2}(u_{n+1}^k - 2u_n^k + u_{n-1}^k) + 2u_n^k - u_n^{k-1} \tag{B1}$$

which is a recursion relation ($k \geq 1$).




To make this method to be stable, we need the square root of the coefficient, $\dfrac{c\Delta t}{\Delta x} \leq 1$.
Then what we have to is apply the initial condition and boundary condition.
The initial condition is usually $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. With discretization, we have

$$u_n^0 = f(n\Delta x)$$

and

$$\frac{u(x,\Delta t) - u(x,-\Delta t)}{2\Delta t} = g(x)$$

which implies

$$u_n^1 - u_n^{-1} = 2\Delta t g(n\Delta x)$$

Plugging into (B1) implies

$$u_n^1 = \frac{1}{2}\frac{c^2 \Delta t^2}{\Delta x^2}(u_n^1 - 2u_n^0 + u_{n-1}^0) + u_n^0 + \Delta t g(n\Delta x) \tag{B2}$$

which gives the recursion relation for $k = 0$.
The boundary conditions usually have two types. If the boundary condition is Dirichlet $u(0,t) = u(L,t) = 0$, we have

$$u_N^k = u_0^k = 0$$

If the boundary condition is Neumann $u_x(0,t) = u_x(L,t) = 0$, with discretization by forward/backward discretization

$$\frac{u(\Delta x, t) - u(-\Delta x, t)}{2\Delta x} = \frac{u(L + \Delta x, t) - u(L - \Delta x, t)}{2\Delta x} = 0$$

which implies

$$u_1^k = u_{-1}^k,\ u_{N+1}^k = u_{N-1}^k$$

Plugging the Neumann boundary condition into (B1) and (B2) would gives

$$u_0^{k+1} = \frac{c^2\Delta t^2}{\Delta x^2}(2u_1^k - 2u_0^k) + 2u_0^k - u_0^{k-1}$$

$$u_N^{k+1} = \frac{c^2\Delta t^2}{\Delta x^2}(2u_{N-1}^k - 2u_N^k) + 2u_N^k - u_N^{k-1}$$

where $k \geq 1$, and

$$u_0^1 = \frac{1}{2}\frac{c^2\Delta t^2}{\Delta x^2}(2u_1^1 - 2u_0^0) + u_0^0 + \Delta t g(0)$$

$$u_N^1 = \frac{1}{2}\frac{c^2\Delta t^2}{\Delta x^2}(2u_{N-1}^1 - 2u_0^0) + u_N^0 + \Delta t g(L)$$

where $k = 0$. In summary, if the boundary condition is Dirichlet, we could write the recursion in matrix-vector product as follows. For $k \geq 1$,

$$\begin{pmatrix} u_0^{k+1} \\ u_1^{k+1} \\ \vdots \\ u_{N-1}^{k+1} \\ u_N^{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 \\ r^2 & 2(1-r^2) & r^2 & \cdots & & 0 \\ & & & \ddots & \ddots & \\ & & & r^2 & 2(1-r^2) & r^2 \\ & & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0^k \\ u_1^k \\ \vdots \\ u_{N-1}^k \\ u_N^k \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 0 \end{pmatrix} \begin{pmatrix} u_0^{k-1} \\ u_1^{k-1} \\ \vdots \\ u_{N-1}^{k-1} \\ u_N^{k-1} \end{pmatrix}$$

for $k = 0$,

$$\begin{pmatrix} u_0^1 \\ u_1^1 \\ \vdots \\ u_{N-1}^1 \\ u_N^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 \\ \frac{1}{2}r^2 & (1-r^2) & \frac{1}{2}r^2 & \cdots & & 0 \\ & & & \ddots & \ddots & \\ & & & \frac{1}{2}r^2 & (1-r^2) & \frac{1}{2}r^2 \\ & & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0^0 \\ u_1^0 \\ \vdots \\ u_{N-1}^0 \\ u_N^0 \end{pmatrix} + \begin{pmatrix} \Delta t g(0) \\ \Delta t g(\Delta x) \\ \vdots \\ \Delta t g((N-1)\Delta x) \\ \Delta t g(N\Delta x) \end{pmatrix}$$

To solve the equation, we need to iterate $k$ from 0 to $K$ while applying the recursion equation on initial condition repeatedly.

4. **Laplace and Poisson Equation**

Since Laplace equation could be treated as a special case of Poisson equation, we are going to use Poisson equation to do the derivation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = S(x, y),\ 0 < x < X,\ 0 < y < Y$$

Discretize it with centered difference,

$$\frac{u(x+\Delta x,y)-2u(x,y)+u(x-\Delta x,y)}{\Delta x^2}+\frac{u(x,y+\Delta y)-2u(x,y)+u(x,y-\Delta y)}{\Delta y^2}=S(x,y)$$

As in Heat equation and wave equation, we could use the index representation

$$u_{n,m}=u(n\Delta x,m\Delta y),\ S_{n,m}=S(n\Delta x,m\Delta y)$$

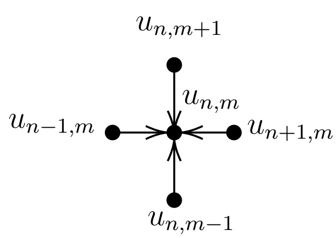
where $0\leq n\leq \dfrac{X}{\Delta x}=N$ and $0\leq m\leq\dfrac{Y}{\Delta y}=M$. Then the equation could be written as

$$\frac{u_{n+1,m}-2u_{u,m}+u_{n-1,m}}{\Delta x^2}+\frac{u_{n,m+1}-2u_{n,m}+u_{n,m-1}}{\Delta y^2}=S_{n,m}$$

Let $\Delta x=\Delta y$, we could pull out $u_{n,m}$,

$$u_{n,m}=\frac{1}{4}(u_{n+1,m}+u_{n-1,m}+u_{n,m+1}+u_{n,m-1}-S_{n,m}\Delta x^2),\ 0<n<N,\ 0<m<M$$

as recursion equation.




Note this equation shows that the solution has the property that the value at each point $(x_n,y_m)$ is the "average" of the values at its four neighbouring points. However, this relationship shows that two neighbor points are dependent on each other, which could not give the solution. We are going to use Jacobi Method based on the property of Laplace/Poisson equation that it is steady. First we have to make a guess of all the $u_{n,m}$ as $u^0_{n,m}$. Then we have to do the iteration

$$u^{k+1}_{n,m}=\frac{1}{4}(u^k_{n+1,m}+u^k_{n-1,m}+u^k_{n,m+1}+u^k_{n,m-1}-S_{n,m}\Delta x^2) \tag{C}$$

until the change between $u^k_{n,m}$ and $u^{k+1}_{n,m}$ is small enough. The tolerance could be calculated as

$$\ell=\sqrt{\frac{1}{MN}\sum_{n=0}^{N}\sum_{m=0}^{M}(u^{k+1}_{n,m}-u^k_{n,m})^2}$$

The initial guess requires boundary condition and it also works for the iteration (Since there is no time involving, there is no initial conditions). The boundary conditions usually are

$$u(0,y)=0,\ u(X,y)=0,\ u(x,Y)=0,\ u(x,0)=f(x)$$

which could be discretized as

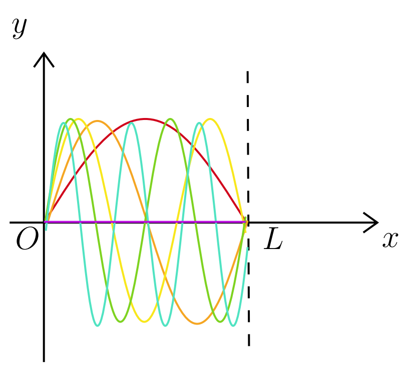
$$u(0,m) = u(N,m) = u(n,M) = 0,\ u(n,0) = f(n\Delta x)$$

Neumann boundary conditions can be incorporated by calculating values for $u_{-1,m}$ (for instance), as for the heat equation.

# Fourier Series

1. **Motivation**

   Consider a heat equation
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L$$
   with Dirichlet boundary condition
$$u(0,t) = u(L,t) = 0$$
   and initial condition
$$u(x,0) = f(x)$$
   We know, from previous section, that a possible solution is
$$u(x,t) = e^{ikx-\alpha^2 k^2 t}$$
   which in the real form is
$$u(x,t) = e^{-\alpha^2 k^2 t}(A\cos(kx) + B\sin(kx))$$
   Apply the boundary condition
$$A = A\cos(kL) + B\sin(kL) = 0$$
   which implies $A = 0$, $B \in \mathbb{R}$ and $k_n = \dfrac{\pi n}{L}$, $n \in \mathbb{Z}$. Therefore the solution could be in the form of
$$u_n(x,t) = B_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \sin\left(\frac{n\pi x}{L}\right)$$
   And the general solution could be the linear combination (superposition)
$$u(x,t) = \sum_{n=0}^{\infty} B_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \sin\left(\frac{n\pi x}{L}\right)$$
   To find out the coefficients, we need to apply the initial condition
$$u(x,0) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$
   Then we are facing an problem which is to represent an arbitrary function in terms of infinite sine/cosine functions.



2. **Family of Cosine Functions**
We define the collection of sine/cosine functions

$$\begin{aligned}\varphi &= \{1, \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \dots\} \\ &= \{1, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\},\ n \in \mathbb{N}_+\end{aligned}$$

as the family of cosine functions. To study how to decompose/expand an arbitrary function $f(x)$ to the base functions in the family of cosine functions, we need to explore two properties of the family.

- Any functions in the family is periodic with the period $T = 2L$.
*Proof.* It is trivial that the constant function $f(x) = 1$ is periodic with any periods. Besides, we have

$$\sin\left(\frac{n\pi(x+2L)}{L}\right) = \sin\left(\frac{n\pi x}{L} + 2n\pi\right) = \sin\left(\frac{n\pi x}{L}\right)$$

and

$$\cos\left(\frac{n\pi(x+2L)}{L}\right) = \cos\left(\frac{n\pi x}{L} + 2n\pi\right) = \cos\left(\frac{n\pi x}{L}\right)$$

which shows $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{n\pi x}{L}\right)$ are periodic with a period $T = 2L$.
*Note.* This property shows that the function $f(x)$ should also be periodic with a period $T = 2L$.

- The family is orthogonal.
*Proof.* Although we don't know what kind of vector space it is here, but we will figure it out as long as we prove the completeness and convergence of linear combination of the base functions in the family. We just need to define the inner product of the family first.

$$\langle f, g \rangle := \int_{-L}^{L} f(x)g(x)\,dx$$

To prove the the orthogonality, we need to prove the inner product of two base functions equals to zero but the inner product of a base function itself is non-zero. We have the combination with 1

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\,dx = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} = 0 \tag{a}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} = 0 \tag{b}$$

and the combination with sine and cosine where $n \neq m$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)\, dx = \int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right] dx$$
$$= 0 \text{ by (b)} \tag{c}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)\, dx = \int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{(n-m)\pi x}{L}\right) + \cos\left(\frac{(n+m)\pi x}{L}\right)\right] dx$$
$$= 0 \text{ by (b)} \tag{d}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)\, dx = \int_{-L}^{L} \frac{1}{2}\left[\sin\left(\frac{(n-m)\pi x}{L}\right) + \sin\left(\frac{(n+m)\pi x}{L}\right)\right] dx$$
$$= 0 \text{ by (a)} \tag{e}$$

and those combination with a sine/cosine itself where $n = m$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)^2 = \int_{-L}^{L} \frac{1+\cos\left(\frac{2n\pi x}{L}\right)}{2}\, dx = L \text{ by (b)} \tag{f}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)^2 = \int_{-L}^{L} \frac{1-\cos\left(\frac{2n\pi x}{L}\right)}{2}\, dx = L \text{ by (b)} \tag{f}$$

and the inner product of a constant function itself

$$\int_{-L}^{L} 1^2\, dx = 2L \tag{h}$$

In summary, we have the family is orthogonal with

$$\int_{-L}^{L} 1^2 dx = 2L \tag{*}$$

and

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)^2 = \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)^2 = L \tag{**}$$

*Note.* This property gives us a hint to use the analogy of projection.

3. **Fourier Series**

   (a) **Definition**

   Assume a periodic function $f(x)$ is able to be expanded to the form

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

The coefficient on $a_0$ comes from the comparison between (*) and (**). Then we want to project $f(x) = F(x)$ on base functions,

$$\int_{-L}^{L} 1 \cdot f(x)\,dx = \int_{-L}^{L} 1 \cdot F(x) = \frac{a_0}{2} \int_{-L}^{L} 1^2\,dx = a_0 L$$

by (*) and

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx = \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) F(x)\,dx = a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)^2\,dx = a_n L$$

by (**) and

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx = \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) F(x)\,dx = b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)^2\,dx = b_n L$$

by (**). Since $1 = \cos\left(\dfrac{0 \times \pi x}{L}\right)$, we could combine $a_0$ and $a_n$ terms. Therefore we could have the definition.

**Definition 1.** Let $f(x)$ to be a periodic function with period $T = 2L$, the series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

is the *Fourier Series* of function $f(x)$ if

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\,dx$$

and

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\,dx$$

Since $f(x)$ is periodic with $T = 2L$, the Fourier series will be the same if $f(x)$ is shifted by $2L$. Besides, there is one thing to notice: we DON'T know whether $F(x)$ converges to $f(x)$ yet. We have to check the completeness of the base functions and the convergence of $F(x)$.

(b) **Completeness**

In this section we are going to prove that the set of base functions is complete for a periodic continuous function $f(x)$.

**Theorem 1.** (Completeness Theorem) A continuous periodic function $f(x)$ equals its Fourier series.

(c) **Convergence**

Based on the completeness of basis, we could analyze the convergence of $F(x)$ of $f(x)$ with different types of continuity.

**Theorem 2.** (Dirichlet Condition) If $f(x)$ is periodic on a period $T = 2L$ and

- is absolutely integrable over a period;
- is of bounded variation in any given bounded interval;
- has a finite number of discontinuities in any given bounded interval, and the discontinuities cannot be infinite;

then the Fourier Series $F(x)$ of $f(x)$ converges to the value

$$F(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}$$

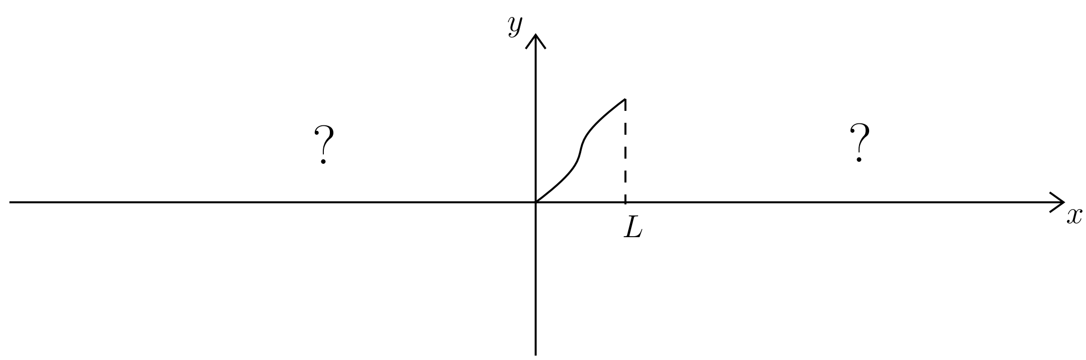
For convenience, we could simplify the condition of $f(x)$.

**Theorem 3.** (Pointwise Convergence) If $f(x)$ and $f'(x)$ are piecewise continuous, then the Fourier Series $F(x)$ of $f(x)$ converges to the value

$$F(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}$$

(d) **Extension Function**

If we are given an non-periodic function $f(x)$ on the domain $[0, L]$, we have to find a path to extend $f(x)$ to be periodic on $\mathbb{R}$ with a period $T = 2\pi$ so that we can expand $f(x)$ to a Fourier Series. There are basically three ways to extend it.



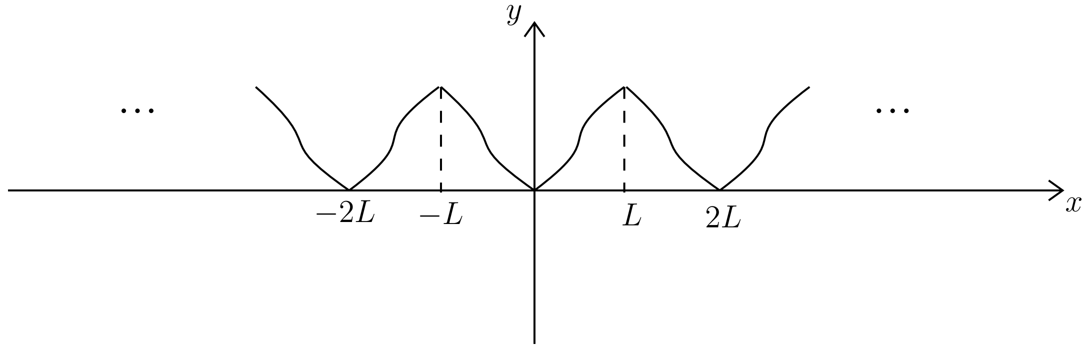

- Half Range Even Extension: If we know the expanded Fourier series is a cosine series

$$C(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

we would be sure that $f(x)$ is an even function. Then we extend $f(x)$ to $[-L, 0]$ first by $f(-x) = f(x)$. Then we extend the domain $[-L, L]$ to the full range of $\mathbb{R}$ with the periodic property $f(x + 2L) = f(x)$. Notice the convergence should be checked based on the continuity of the extended functions.



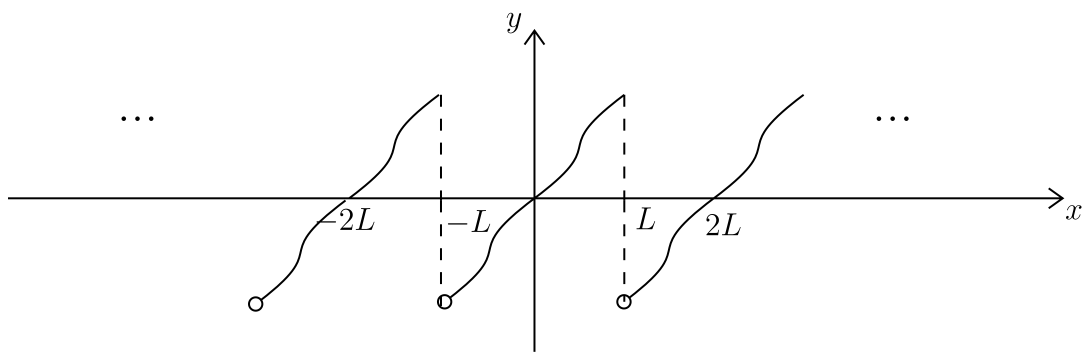

- Half Range Odd Extension: If we know the expanded Fourier series is sine series

$$S(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

we would be sure that $f(x)$ is an odd function. Then we extend $f(x)$ to $[-L, 0]$ first by $f(-x) = -f(x)$. Then we extend the domain $[-L, L]$ to the full range of $\mathbb{R}$ with the periodic property $f(x + 2L) = f(x)$. Notice the convergence should be checked based on the continuity of the extended functions.




- Full Range Extension: If there is no enough information given, we could use $[0, L]$ as the whole period with $f(x + L) = f(x)$. Then the series becomes

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right)$$

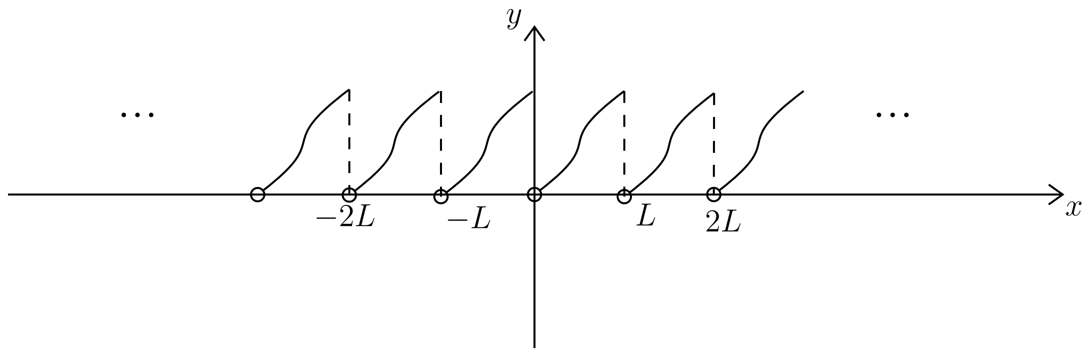
where

$$a_n = \frac{2}{L}\int_{0}^{L} f(x) \cos\left(\frac{2n\pi x}{L}\right)\, dx$$

and

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2n\pi x}{L}\right) dx$$

Notice the convergence should be checked based on the continuity of the extended functions.




(e) **Approximation and Error Analysis**
Consider the Fourier Series $F(x)$ of function $f(x)$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) = S_\infty(x)$$

We want to analyze the approximation of $f(x)$ with the Fourier Series. Denote

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Correspondingly, we define the error as analogy of the normalized square of distance in the vector distance, which would be least-square error

$$\varepsilon_2[f, S_N] = \frac{1}{L} \langle f - S_N, f - S_N \rangle$$

Then we have

$$\begin{aligned}
\varepsilon_2[f, S_N] &= \frac{1}{L} \int_{-L}^{L} [f(x) - S_N(x)]^2 dx \\
&= \frac{1}{L} \int_{-L}^{L} f(x)^2 dx + \frac{1}{L} \int_{-L}^{L} S_N^2(x) dx - \frac{2}{L} \int_{-L}^{L} f(x) S_N(x) dx \\
&= \frac{1}{L} \int_{-L}^{L} f(x)^2 dx + \frac{1}{L} \left( \frac{a_0^2}{2} L + \sum_{n=1}^{N} (a_n^2 L + b_n^2 L) \right) - \frac{2}{L} \left( \frac{a_0^2}{2} L + \sum_{n=1}^{N} (a_n^2 L + b_n^2 L) \right) \\
&= \frac{1}{L} \langle f, f \rangle - \left( \frac{a_0^2}{2} + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right)
\end{aligned}$$

With the square, we know $\varepsilon_2[f, S_N] \geq 0$. Then we know

$$\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2) \leq \frac{1}{L}\int_{-L}^{L} f(x)^2 dx$$

If the integral $\frac{1}{L}\int_{-L}^{L} f(x)^2 dx < \infty$ is finite, the series could be bounded. Then we denote $f \in L_2[-L, L]$ define the bound to be the energy of $2L$-periodic function $f(x)$, $E[f] = \frac{1}{L}\int_{-L}^{L} f(x)^2 dx$. Take the limit, we have Bessel's Inequality.

**Theorem 4.** (Bessel's Inequality) If $f \in L_2[-L, L]$, then with the Fourier series of $f(x)$,

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{L}\int_{-L}^{L} f(x)^2 dx$$

in particular the series $\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)$ is convergent.

If this series converges to the energy of the function, we have our least-square error $\varepsilon_2[f, S_\infty] = 0$. Then it is possible that our Fourier series converges to the original function. This is demonstrated by the Parseval's Theorem.

**Theorem 5.** (Parseval's Theorem) Let $f(x) \in L_2[-L, L]$ then the Fourier coefficients $a_n$ and $b_n$ satisfy Parseval's Formula

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} f(x)^2 dx$$

if and only if

$$\lim_{N\to\infty}\int_{-L}^{L}[f(x) - S_N(x)]^2\, dx = 0$$

In the case we know the least square error is

$$\varepsilon_2[f(x), S_N(x)] = \sum_{n=N+1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Here we are going to look at a few examples.

**Example 11.** (Proof of Parseval's Identity for odd functions)

For odd continuous periodic functions, we know $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$. Then we want to prove

$$\frac{1}{L}\int_{-L}^{L} f(x)^2\, dx = \sum_{n=1}^{\infty} b_n^2$$

We have to expand the integral

$$
\begin{aligned}
&\frac{1}{L}\int_{-L}^{L}\left(\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\right)\left(\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\right) dx \\
=&\frac{1}{L}\int_{-L}^{L}\sum_{n=1}^{\infty} b_n^2 \sin\left(\frac{n\pi x}{L}\right)^2 dx \\
=&\frac{1}{L}\times L\sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} b_n^2
\end{aligned}
$$

which proves the identity.

**Example 12.** (Gibbs Phenomenon)
The truncated Fourier series near a jump discontinuity overshoots the jump by about 9% of the size of the jump. We are going to demonstrate this on a square wave

$$
f(x) = \begin{cases} 1, \text{ if } x \in (n\pi, (n+1)\pi], n \text{ is even} \\ -1, \text{ if } x \in (n\pi, (n+1)\pi], n \text{ is odd} \end{cases}
$$

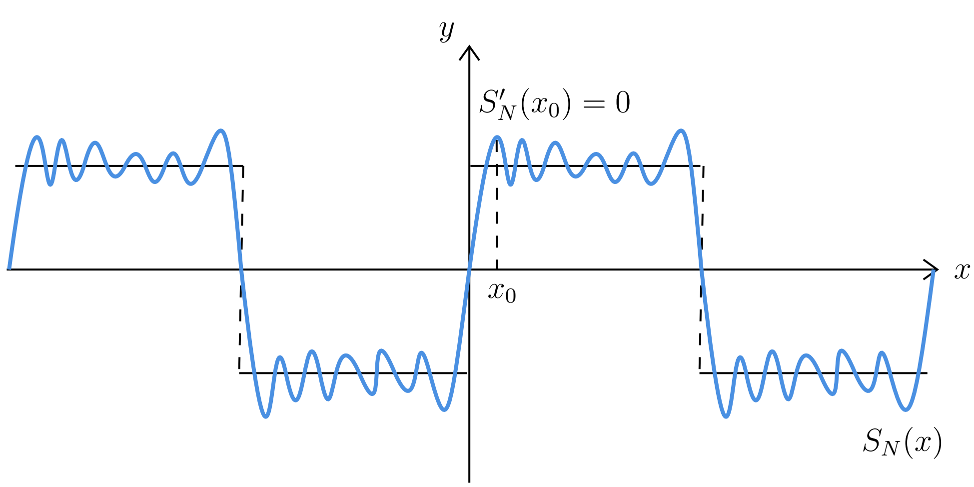
We know the Fourier series of $f(x)$ is

$$
F(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x)}{2n-1}
$$

Then the approximation of $f(x)$ with $N$ terms would be

$$
S_N(x) = \frac{4}{\pi}\sum_{n=1}^{N}\frac{\sin((2n-1)x)}{2n-1}
$$

Then to see the overshoots, we want to find the highest point $S_N(x)$ could go.



To do this, we need to find the derivative of $S_N(x)$. We have

$$S_N'(x) = \frac{4}{\pi}\sum_{n=1}^{N}\cos((2n-1)x)$$

We are going to play some trick to simplify $S_N'(x)$. Multiply $\sin(x)$ and divide it on $S_N'(x)$, we have

$$\begin{aligned}
S_N'(x) &= \frac{4}{\pi\sin(x)}\left(\cos(x)\sin(x) + \cos(3x)\sin(x) + \cdots + \cos((2N-1)x)\sin(x)\right)\\
&= \frac{4}{\pi\sin(x)}\left(\frac{\cancel{\sin(2x)} - \sin(0) + \cancel{\sin(4x)} - \cancel{\sin(2x)} + \cdots + \sin((2N)x) - \cancel{\sin((2N-2)x)}}{2}\right)\\
&= \frac{2}{\pi}\frac{\sin(2Nx)}{\sin(x)}
\end{aligned}$$

Let $S_N'(x_0) = 0$, we have $x = \dfrac{k\pi}{2N}$ where $k \in \mathbb{Z}$. We want $x_0$ to be the first maximal point, so $x_0 = \dfrac{\pi}{2N}$. Then we plug $x_0$ back to $S_N(x)$, we have

$$S_N\left(\frac{\pi}{2N}\right) = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{2N}\right) + \frac{\sin\left(\frac{3\pi}{2N}\right)}{3} + \cdots + \frac{\sin\left(\frac{(2N-1)\pi}{2N}\right)}{2N-1}\right)$$

We could try few terms to see what is going on.

- $N = 1$, we know $S_1(\dfrac{\pi}{2}) = 1.273$ and then the percentage of overshoot is

$$\eta = \frac{1.273 - 1}{2} = 13.6\%$$

- $N = 21$, we know $S_{21}(\dfrac{\pi}{42}) = 1.178$ and then the percentage of overshoot is

$$\eta = \frac{1.178 - 1}{2} = 9\%$$

It seems that taking more terms giving more accurate answer. Then we are going to analytically calculate the approximation. We have

$$S_N\left(\frac{\pi}{2N}\right) = \frac{2}{\pi}\frac{\pi}{N}\left(\frac{\sin(\pi/2N)}{\pi/2N} + \frac{\sin(3\pi/2N)}{3\pi/2N} + \cdots + \frac{\sin((2N-1)\pi/2N)}{(2N-1)\pi/2N}\right)$$

which could be treated as the Riemann sum of the integral

$$\frac{\pi}{2}\int_0^{\pi}\frac{\sin(x)}{x}\,dx,\ \Delta x = \frac{\pi}{N}$$

Therefore

$$\lim_{N\to\infty} S_N\left(\frac{\pi}{2N}\right) = \frac{\pi}{2}\int_0^{\pi}\frac{\sin(x)}{x}\,dx$$

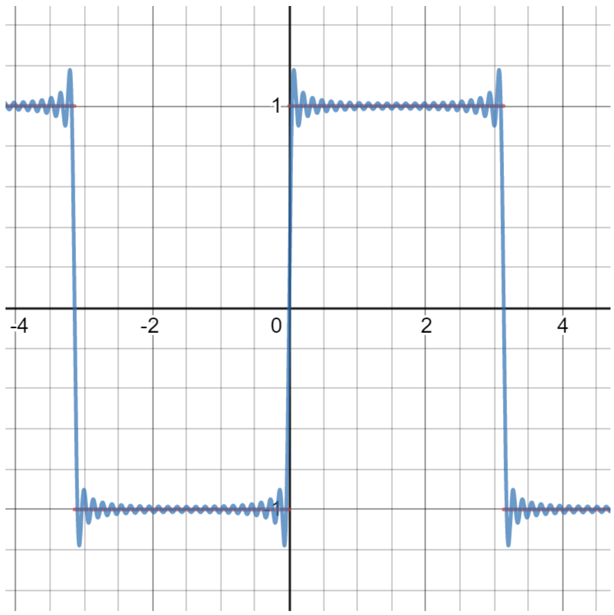
Then we could use the integral of the power series $\frac{\pi}{2}\int_0^{\pi}\frac{\sin(x)}{x}\,dx$ to estimate $S_N(x)$ if we have enough $N$ terms. Therefore

$$S_N(x) \approx 2\left(1 - \frac{\pi^2}{3\cdot 3!} + \frac{\pi^4}{5\cdot 5!} - \frac{\pi^6}{7\cdot 7!} + \frac{\pi^8}{9\cdot 9!} - \cdots\right) \approx 1.18$$

And the percentage of overshoot is

$$\eta = \frac{1.18 - 1}{2} = 9\%$$

Here we attached a graph calculated by demos with $N = 18$




On the other hand, we could also use the Dirichlet condition and Parseval's identity to calculate some series.

**Example 13.** ($p$-series)
Consider the continuous function $f(x) = x$, $x \in [0, 2]$, which, with odd extension, could be expanded as

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{2}\right)$$

The energy is

$$E[f] = \frac{2}{2}\int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

and we have

$$\sum_{n=1}^{\infty} b_n^2 = \frac{4^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

By Parseval's identity, we have

$$\frac{8}{3} = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Therefore we know

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Consider the continuous function $f(x) = x^2$, $x \in (-\pi, \pi)$ which could be expanded as

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

With

$$\frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \frac{\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

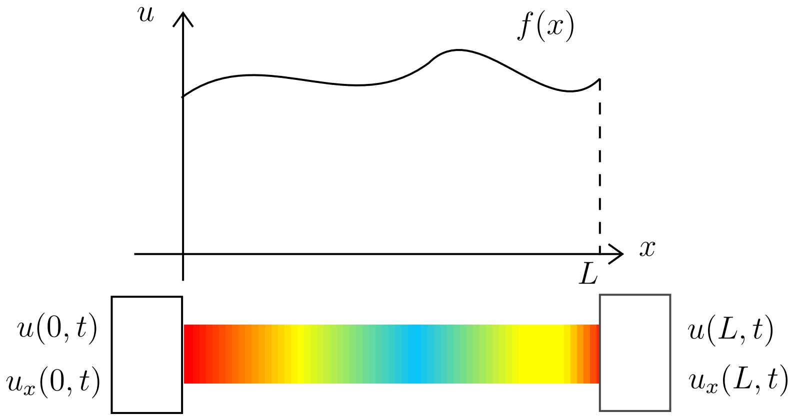
we know

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{90}$$

## Method of PDEs: Separation of Variables

1. **Heat Equation**

   (a) **Standard Heat Conduction Equation - Heat Conduction in a Rod**
   Consider the heat conduction in a rod ($0 < x < L$),




with the initial condition $u(x, 0) = f(x)$ and some boundary conditions of $u(0, t)$, $u(L, t)$, $u_x(0, t)$ and $u_x(L, t)$. If the conduction only happens in the rod and with the end, we know the differential equation governing the conduction is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L$$

To solve this equation, we are going to use the technique called *separation of variables*. First we make a *Fourier's guess* that the solution

$$u(x, t) = X(x)T(t)$$

is separable. Then we have

$$u_t(x, t) = X(x)T'(t),\ u_x(x, t) = X'(x)T(t),\ u_{xx}(x, t) = X''(x)T(t)$$

Plug them back to the equation, we have

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Rearrange the equation, we have

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{No } t \text{ terms!}} = \underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{No } x \text{ terms!}} = \underbrace{\text{constant}}_{\text{No } x \text{ and } t \text{ terms!}} = \lambda$$

Then we have get two separate differential equations

$$T' - \lambda\alpha^2 T = 0 \qquad \text{(A)}$$

and

$$X'' - \lambda X = 0 \qquad \text{(B)}$$

The first equation (A) could be solved regardless of the value of $\lambda$,

$$T(t) = e^{\lambda \alpha^2 t}$$

But the second equation should be solved based of the value of $\lambda$. To restrict the number of value and solve for $X(x)$, we need to apply the boundary condition.

i. **Dirichlet Boundary Condition**
If two ends are placed with ice with 0°C,


then we have the Dirichlet boundary condition

$$u(0,t) = u(L,t) = 0$$

Separate the boundary condition, we have

$$X(0)T(t) = X(L)T(t) = 0$$

Since $T(t) \neq 0$, we could cancel the $T(t)$ to get

$$X(0) = X(L) = 0$$

Based on the boundary condition, we could discuss the choice of $\lambda$ to get a non-trivial solution of $X(x)$.

If $\lambda > 0$, we let $\lambda = +\mu^2$, $\mu > 0$. Then the differential equation is $X'' - \mu^2 X = 0$ with the characteristic equation $r^2 - \mu^2 = 0$ implying $r = \pm\mu$. Then we have the solution to be

$$X(x) = Ae^{\mu x} + Be^{-\mu x}$$

Plug into the boundary condition, we have

$$A + B = Ae^{\mu L} + Be^{-\mu L} = 0$$

Since $e^{\mu L}, e^{-\mu L} \neq 0$, we have $A = B = 0$ which gives $X(x) = 0$, a trivial solution.

If $\lambda = 0$, then the differential equation is $X'' = 0$ and the solution is $X(x) = Ax + B$. Plug into the boundary condition, we have

$$B = AL + B = 0$$

which implies $A = B = 0$ and $X(x) = 0$, a trivial solution.

If $\lambda < 0$, let $\lambda = -\mu^2$, $\mu > 0$. Then the differential equation is $X'' + \mu^2 X = 0$ with the characteristic equation $r^2 + \mu^2 = 0$ implying $r = \pm i\mu$. Then we have the solution to be

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

Plug into the boundary condition, we have

$$A = A\cos(\mu L) + B\sin(\mu L) = 0$$

It shows that $A = 0$. We don't want $B = 0$ which would give another trivial solution, so we have $\mu L = n\pi$. Then we have

$$\boxed{X_n(x) = \sin\left(\frac{n\pi x}{L}\right),\ \lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2,\ n \in \mathbb{Z}^+} \tag{i}$$

Here $n \neq 0$ is to guarantee that $X(x)$ in non-trivial and $n > 0$ is to make sure $X_n$ is not repeated as sin function is odd.

Then we have

$$u_n(x,t) = e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)$$

and with the principle of superposition, the complete solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)$$

Then to find the coefficient $B_n$, we need to apply the initial condition,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = S(x)$$

This requires us to expand $f(x)$ as an Fourier sine series. First we have to (odd) extant the f(x) to the domain $-L < x < L$ with $f(x) = -f(-x)$ and then we extent it to the entire domain $\mathbb{R}$. Then with

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

we could find all the coefficients. Then the solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right)e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)$$

ii. **Neumann Boundary Condition**

If two ends are isolated,


then we have the Neumann boundary condition

$$u_x(0,t) = u_x(L,t) = 0$$

Separate the boundary condition, we have

$$X'(0)T(t) = X'(L)T(t) = 0$$

Since $T(t) \neq 0$, we could cancel the $T(t)$ to get

$$X'(0) = X'(L) = 0$$

Based on the boundary condition, we could discuss the choice of $\lambda$ to get a non-trivial solution of $X(x)$.

If $\lambda > 0$, we let $\lambda = +\mu^2$, $\mu > 0$. Then the differential equation is $X'' - \mu^2 X = 0$ with the characteristic equation $r^2 - \mu^2 = 0$ implying $r = \pm\mu$. Then we have the solution to be

$$X(x) = Ae^{\mu x} + Be^{-\mu x}$$

Plug into the boundary condition, we have

$$A\mu + B\mu = A\mu e^{\mu L} - B\mu e^{-\mu L} = 0$$

Since $e^{\mu L}, e^{-\mu L} \neq 0$, we have $A = B = 0$ which gives $X(x) = 0$, a trivial solution.

If $\lambda = 0$, then the differential equation is $X'' = 0$ and the solution is $X(x) = Ax+B$. Plug into the boundary condition, we have

$$A = A = 0$$

which implies $A = 0, B \in \mathbb{R}$ and $X(x) = 1$ with $\lambda = 0$.

If $\lambda < 0$, let $\lambda = -\mu^2$, $\mu > 0$. Then the differential equation is $X'' + \mu^2 X = 0$ with the characteristic equation $r^2 + \mu^2 = 0$ implying $r = \pm i\mu$. Then we have the solution to be

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

Plug into the boundary condition, we have

$$\mu B = -\mu A\sin(\mu L) + \mu B\cos(\mu L) = 0$$

It shows that $B = 0$. We don't want $A = 0$ which would give another trivial solution, so we have $\mu L = n\pi$. Then we have

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \ \lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, \ n \in \mathbb{Z}^+$$

Here $n \neq 0$ is to guarantee $\lambda < 0$ and $n > 0$ is to make sure $X_n$ is not repeated as cos function is even.

Combine the situation of $\lambda < 0$ and $\lambda = 0$, we have

$$\boxed{X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \ \lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, \ n \in \mathbb{Z}_0^+} \tag{ii}$$

Then we have

$$u_n(x,t) = e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

and with the principle of superposition, the complete solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

Then to find the coefficient $A_n$, we need to apply the initial condition,

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = C(x)$$

This requires us to expand $f(x)$ as an Fourier cosine series. First we have to (even) extant the f(x) to the domain $-L < x < L$ with $f(x) = f(-x)$ and then we extent it to the entire domain $\mathbb{R}$. Then with

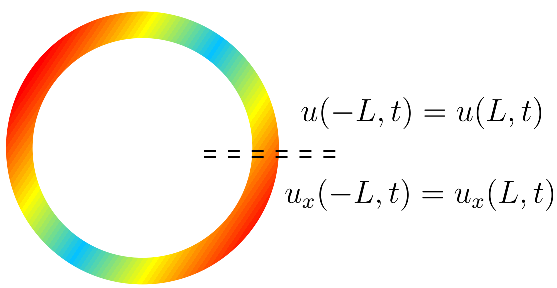
$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx$$

we could find all the coefficients. Then the solution is

$$u(x,t) = \sum_{n=0}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

iii. **Periodic Boundary Condition**

If the bar is a circle




then we have the Periodic boundary condition

$$u(0,t) = u(L,t),\ u_x(0,t) = u_x(L,t)$$

Separate the boundary condition, we have

$$X(0)T(t) = X(L)T(t),\ X'(0)T(t) = X'(L)T(t)$$

Since $T(t) \neq 0$, we could cancel the $T(t)$ to get

$$X(0) = X(L),\ X'(0) = X'(L)$$

Based on the boundary condition, we could discuss the choice of $\lambda$ to get a non-trivial solution of $X(x)$.

If $\lambda > 0$, we let $\lambda = +\mu^2$, $\mu > 0$. Then the differential equation is $X'' - \mu^2 X = 0$ with the characteristic equation $r^2 - \mu^2 = 0$ implying $r = \pm\mu$. Then we have the solution to be

$$X(x) = Ae^{\mu x} + Be^{-\mu x}$$

Plug into the boundary condition, we have

$$A + B = Ae^{\mu L} + Be^{-\mu L},\ A\mu - B\mu = A\mu e^{\mu L} - B\mu e^{-\mu L}$$

Since $e^{\mu L}, e^{-\mu L} \neq 0$, we have $A = B = 0$ which gives $X(x) = 0$, a trivial solution.

If $\lambda = 0$, then the differential equation is $X'' = 0$ and the solution is $X(x) = Ax+B$. Plug into the boundary condition, we have

$$B = AL + B,\ A = A$$

which implies $A = 0, B \in \mathbb{R}$ and $X_0(x) = 1$ with $\lambda_0 = 0$.

If $\lambda < 0$, let $\lambda = -\mu^2$, $\mu > 0$. Then the differential equation is $X'' + \mu^2 X = 0$ with the characteristic equation $r^2 + \mu^2 = 0$ implying $r = \pm i\mu$. Then we have the solution to be

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

Plug into the boundary condition, we have

$$A\cos(\mu L) - B\sin(\mu L) = A\cos(\mu L) + B\sin(\mu L)$$

$$\mu A\sin(\mu L) + \mu B\cos(\mu L) = -\mu A\sin(\mu L) + \mu B\cos(\mu L)$$

We don't want $A = 0, B = 0$ which would give another trivial solution, so we have $\mu L = n\pi$. Then we have

$$X_n(x) = \left\{\cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\},\ \lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2,\ n \in \mathbb{Z}^+$$

Here $n \neq 0$ is to guarantee $\lambda < 0$ and $n > 0$ is to make sure $X_n$ is not repeated as cos function is even.

Combine the situation of $\lambda < 0$ and $\lambda = 0$, we have

$$\boxed{X_n(x) = \left\{1,\ \cos\left(\frac{n\pi x}{L}\right),\ \sin\left(\frac{n\pi x}{L}\right)\right\},\ \lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2,\ n \in \mathbb{Z}_0^+} \tag{iii}$$

Then we have

$$u_n(x,t) \in \left\{1,\ e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right),\ e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)\right\}$$

and with the principle of superposition, the complete solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty}\left(A_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right) + B_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)\right)$$

Then to find the coefficient $A_n$, we need to apply the initial condition,

$$f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right) = F(x)$$

This requires us to expand $f(x)$ as a Fourier series. We extent it to the entire domain $\mathbb{R}$. Then with

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

and

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

and

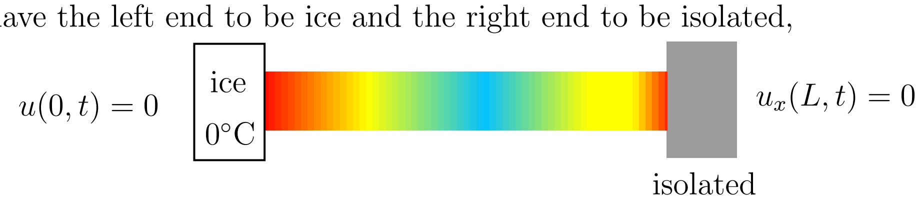
$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

we could find all the coefficients. Then the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ &\quad + \sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right) \\ &\quad + \sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

iv. **Mixed Boundary Condition**

If we have the left end to be ice and the right end to be isolated,




then we have the mixed boundary condition

$$u(0,t) = u_x(L,t) = 0$$

Separate the boundary condition, we have

$$X(0)T(t) = X'(L)T(t) = 0$$

Since $T(t) \neq 0$, we could cancel the $T(t)$ to get

$$X(0) = X'(L) = 0$$

Based on the boundary condition, we could discuss the choice of $\lambda$ to get a non-trivial solution of $X(x)$.

If $\lambda > 0$, we let $\lambda = +\mu^2$, $\mu > 0$. Then the differential equation is $X'' - \mu^2 X = 0$ with the characteristic equation $r^2 - \mu^2 = 0$ implying $r = \pm\mu$. Then we have the solution to be

$$X(x) = Ae^{\mu x} + Be^{-\mu x}$$

Plug into the boundary condition, we have

$$A + B = A\mu e^{\mu L} - B\mu e^{-\mu L} = 0$$

Since $e^{\mu L}, e^{-\mu L} \neq 0$, we have $A = B = 0$ which gives $X(x) = 0$, a trivial solution.

If $\lambda = 0$, then the differential equation is $X'' = 0$ and the solution is $X(x) = Ax+B$. Plug into the boundary condition, we have

$$B = A = 0$$

which implies $A = B = 0$ and $X(x) = 0$, a trivial solution.

If $\lambda < 0$, let $\lambda = -\mu^2$, $\mu > 0$. Then the differential equation is $X'' + \mu^2 X = 0$ with the characteristic equation $r^2 + \mu^2 = 0$ implying $r = \pm i\mu$. Then we have the solution to be

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

Plug into the boundary condition, we have

$$A = -A\mu\sin(\mu L) + B\mu\cos(\mu L) = 0$$

It shows that $A = 0$. We don't want $B = 0$ which would give another trivial solution, so we have $\mu L = \left(n + \frac{1}{2}\right)\pi$. Then we have

$$\boxed{X_n(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right),\ \lambda_n = -\mu_n^2 = -\left(\frac{(2n+1)\pi}{2L}\right)^2,\ n \in \mathbb{Z}_0^+} \tag{iv}$$

Here $n \geq 0$ is to make sure $X_n$ is not repeated as sin function is odd.

Then we have

$$u_n(x,t) = e^{-\left(\frac{(2n+1)\pi\alpha}{2L}\right)^2 t}\sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

and with the principle of superposition, the complete solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{(2n+1)\pi\alpha}{2L}\right)^2 t}\sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

Then to find the coefficient $B_n$, we need to apply the initial condition,

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{(2n+1)\pi x}{2L}\right) = S(x)$$

This requires us to expand $f(x)$ as an Fourier sine series. First we have to (odd) extant the $f(x)$ to the domain $-L < x < L$ with $f(x) = -f(-x)$ and then we extent it to the entire domain $\mathbb{R}$. Then with

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right)\,dx$$

we could find all the coefficients. Then the solution is

$$u(x,t) = \sum_{n=0}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right)\,dx\right)e^{-\left(\frac{(2n+1)\pi\alpha}{2L}\right)^2 t}\sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

If we have the left end to isolated and right end to be ice,

$u_x(0,t) = 0$


$u(L,t) = 0$

isolated

as the opposite boundary condition

$$u_x(0,t) = u(L,t) = 0$$

We could use the property of symmetry,

$$X_{\text{opposite}}(x) = X(L-x) = \sin\left(\frac{(2n+1)\pi}{2} - \frac{(2n+1)\pi x}{2L}\right) = \pm\cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

which implies

$$\boxed{X_n(x) = \cos\left(\frac{(2n+1)\pi x}{2L}\right),\ \lambda_n = -\mu_n^2 = -\left(\frac{(2n+1)\pi}{2L}\right)^2,\ n \in \mathbb{Z}_0^+} \tag{v}$$

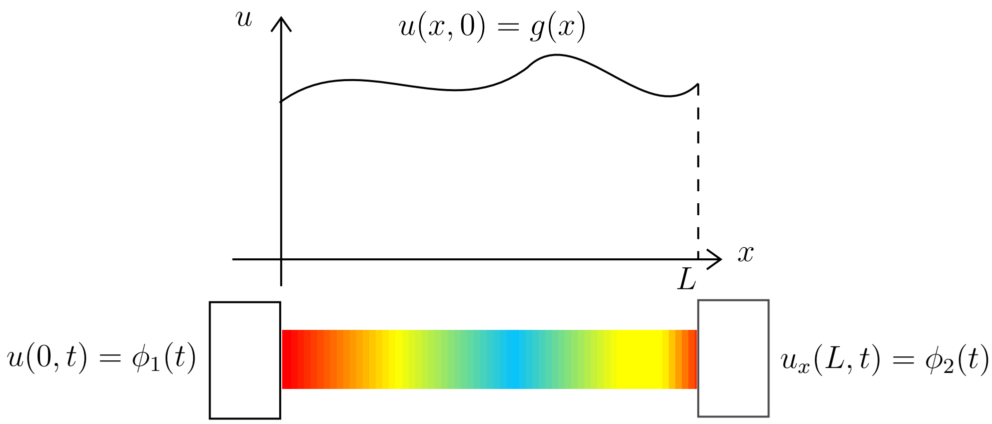
and the solution is

$$u(x,t) = \sum_{n=0}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right)\,dx\right)e^{-\left(\frac{(2n+1)\pi\alpha}{2L}\right)^2 t}\cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

with the even extension of $f(x)$.

(b) **Inhomogeneous Heat Equation**
Generally, inhomogeneous could refer to the boundary condition which is not zero and could refer to the equation with a source or sink as an extra term. And the boundary condition and the extra term could be time-dependent or time-independent.

For example,




$$u_t = \alpha^2 u_{xx} + f(x,t),\ 0 < x < L,\ t > 0$$

The general idea is similar to the inhomogeneous ODEs - we could first find a steady state (or particular) $w(x) = u(x,\infty)$, then after subtracting the particular solution, we could find the transient (homogeneous) solution $v(x,t)$. The general solution would be the sum of particular solution $u(x,t) = w(x) + v(x,t)$.

Let's look at some special cases.

i. **Time-independent Boundary Condition**
If the boundary condition is time-independent, then they should be constant (at least on side is non-zero). They could be Dirichlet, Neumann or mixed and sometimes we need the equation to have an extra term to be solvable.

**Example 14.** (Dirichlet Boundary Condition)
Consider the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0$$

with the inhomogeneous Dirichlet boundary condition

$$u(0,t) = u_1,\ u(L,t) = u_2$$

and the initial condition

$$u(x,0) = f(x)$$

First we want to find the steady state solution. We denote $w(x) = u(x,\infty)$ which does not change with time. Since it is steady, we know

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial w}{\partial t} = 0$$

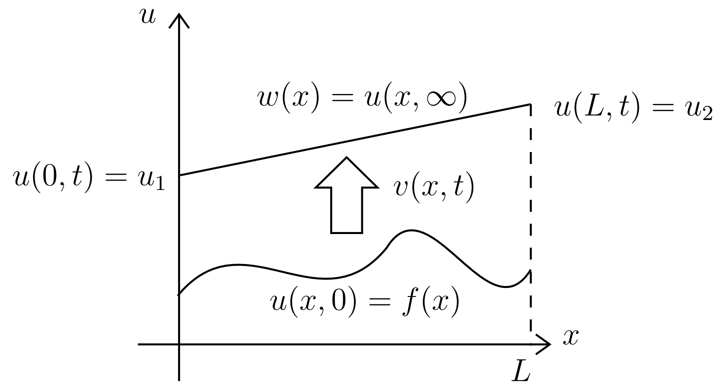
which implies

$$w(x) = Ax + B$$

Plug in the boundary condition $w(0) = u(0, \infty) = u_1$, $w(L) = u(L, \infty) = u_2$, we have $A = \dfrac{u_2 - u_1}{L}$ and $B = u_1$. Therefore, the steady state solution is

$$w(x) = \frac{u_1 - u_1}{L} x + u_1$$

Then we need to find the transient solution which transfer the initial condition to the steady state.




With $v(x, t) = u(x, t) - w(x)$ where $u(x, t)$ is the complete solution, we have the heat equation to be

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}(u(x, t) - w(x)) \\ &= \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} = \alpha^2 \frac{\partial^2}{\partial x^2}(v(x, t) + w(x)) \\ &= \alpha^2 \frac{\partial^2 v}{\partial x^2} \end{aligned}$$

as a standard heat equation

$$\frac{\partial v}{\partial t} = \alpha^2 \frac{\partial^2 v}{\partial x^2},\ 0 < x < L,\ t > 0$$

and the boundary condition to be

$$v(0, t) = u(0, t) - w(0) = u_1 - u_1 = 0$$
$$v(L, t) = u(L, t) - w(L) = u_2 - u_2 = 0$$

and the initial condition to be

$$v(x, 0) = u(x, 0) - w(x) = f(x) - w(x)$$

This turns out to be a homogeneous heat equation with Dirichlet boundary condition $u(0, t) = u(L, t) = 0$ and initial condition $u(x, 0) = f(x) - w(x)$. The solution is

$$v(x, t) = \sum_{n=1}^{\infty} \underbrace{\left( \frac{1}{L} \int_{-L}^{L} (f(x) - w(x)) \sin\left(\frac{n\pi x}{L}\right) \, dx \right)}_{b_n} e^{-\frac{n^2\pi^2\alpha^2}{L^2} t} \sin\left(\frac{n\pi x}{L}\right)$$

Then the complete solution is

$$u(x,t) = \underbrace{\sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)}_{\text{transient, decays to zero}} + \underbrace{\frac{u_1 - u_1}{L}x + u_1}_{\text{steady state solution}}$$

However, when we have the Neumann boundary condition, it is not always possible to find a steady state solution if the flow on two sides is not balanced ($u_x(0,t) \neq u_x(L,t)$). The idea is still to find a particular solution first.

**Example 15.** (Neumann Boundary condition)
Consider the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0$$

with the inhomogeneous Neumann boundary condition

$$u_x(0,t) = q_1,\ u_x(L,t) = q_2$$

and the initial condition

$$u(x,0) = f(x)$$

Based on the solution would not be steady, we guess a particular solution with the form

$$w(x,t) = \underbrace{Ax^2 + Bx + C}_{\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial w}{\partial t} \neq 0} + \underbrace{Dt}_{\frac{\partial w}{\partial t} \neq 0}$$

Plug into the equation we have

$$D = 2\alpha^2 A$$

Plug into the boundary condition

$$w_x(0,t) = B = q_1$$

$$w_x(L,t) = 2AL + B = q_2$$

These solve $B = q_1$, $A = \dfrac{q_2 - q_1}{2L}$ and $D = \dfrac{2\alpha^2(q_2 - q_1)}{2L}$; and $C$ could be an arbitrary constant. Therefore the particular solution is

$$w(x,t) = \frac{q_2 - q_1}{2L}x^2 + q_1 x + \frac{2\alpha^2(q_2 - q_1)}{2L}t + C$$

Then we want to find the rest part of the solution $v(x,t) = u(x,t) - w(x,t)$. Plug it into the equation and boundary condition we have the equation to be

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}(u(x,t) - w(x,t)) \\
&= \frac{\partial u}{\partial t} - \frac{\partial w}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} - \alpha^2 \frac{\partial^2 w}{\partial x^2} \\
&= \alpha^2 \frac{\partial^2 v}{\partial x^2}
\end{aligned}$$

as a standard heat equation

$$\frac{\partial v}{\partial t} = \alpha^2 \frac{\partial^2 v}{\partial x^2},\ 0 < x < L,\ t > 0$$

and the boundary condition to be

$$v_x(0,t) = u_x(0,t) - w_x(0,t) = q_1 - q_1 = 0$$
$$v_x(L,t) = v_x(L,t) - w(x,t) = q_2 - q_2 = 0$$

and the initial condition to be

$$v(x,0) = u(x,0) - w(x,0) = f(x) - w(x,0)$$

This turns out to be a homogeneous heat equation with Neumann boundary condition $v_x(0,t) = v_x(L,t) = 0$ and initial condition $u(x,0) = f(x) - w(x,0)$. The solution is

$$v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} (f(x) - w(x,t)) \cos\left(\frac{n\pi x}{L}\right)\, dx$$

Then the complete solution is

$$u(x,t) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \cos\left(\frac{n\pi x}{L}\right)}_{\text{particular solution}} + \underbrace{\frac{q_2 - q_1}{2L}x^2 + q_1 x + \frac{2\alpha^2(q_2 - q_1)}{2L}t + C}_{\text{homogeneous solution}}$$

We have to check whether $C$ is actually arbitrary. Sum up the constant with variation $\Delta C$,

$$\begin{aligned}
\frac{a_0'}{2} + C + \Delta C &= \frac{1}{2L}\int_{-L}^{L} (f(x) - w(x,t) - \Delta C)\, dx + C + \Delta C \\
&= \frac{1}{2L}\int_{-L}^{L} (f(x) - w(x,t))\, dx - \frac{1}{2L}\int_{-L}^{L} \Delta C\, dx + C + \Delta C \\
&= \frac{a_0}{2} - \Delta C + C + \Delta C \\
&= \frac{a_0}{2} + C
\end{aligned}$$

This means the choice of $C$ does not matter with the constant term in the complete solution, which demonstrates that the $C$ could be arbitrary. Then we could just pick $C = 0$.

When the boundary condition is mixed, it is possible to find the steady solution if one end flow is indicated as 0 ($u_x(0,t) = 0$ or $u_x(L,t) = 0$). However the keypoint is always to find the particular solution.

**Example 16.** (Mixed Boundary Condition)
Consider the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0$$

with the inhomogeneous Mixed boundary condition

$$u(0,t) = u_0,\ u_x(L,t) = 0$$

and the initial condition

$$u(x,0) = f(x)$$

First we want to find the steady state solution since the right flow is zero. We denote $w(x) = u(x,\infty)$ which does not change with time. Since it is steady, we know

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial w}{\partial t} = 0$$

which implies

$$w(x) = Ax + B$$

Plug in the boundary condition $w(0) = u(0,\infty) = u_0$, $w_x(L) = u_x(L,\infty) = 0$, we have $A = 0$ and $B = u_0$. Therefore, the steady state solution is

$$w(x) = u_0$$

Then we need to find the transient solution which transfer the initial condition to the steady state.
With $v(x,t) = u(x,t) - w(x)$ where $u(x,t)$ is the complete solution, we have the heat equation to be

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}(u(x,t) - w(x)) \\ &= \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} = \alpha^2 \frac{\partial^2}{\partial x^2}(v(x,t) + w(x)) \\ &= \alpha^2 \frac{\partial^2 v}{\partial x^2}\end{aligned}$$

as a standard heat equation

$$\frac{\partial v}{\partial t} = \alpha^2 \frac{\partial^2 v}{\partial x^2},\ 0 < x < L,\ t > 0$$

and the boundary condition to be

$$v(0,t) = u(0,t) - w(0) = u_0 - u_0 = 0$$

$$v_x(L,t) = u_x(L,t) - w_x(0) = 0$$

and the initial condition to be

$$v(x,0) = u(x,0) - w(x) = f(x) - w(x) = f(x) - u_0$$

This turns out to be a homogeneous heat equation with Dirichlet boundary condition $v(0,t) = v_x(L,t) = 0$ and initial condition $u(x,0) = f(x) - u_0$. The solution is

$$v(x,t) = \sum_{n=0}^{\infty} b_n e^{-\frac{(2n+1)^2\pi^2\alpha^2}{4L^2}t} \sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

where

$$b_n = \frac{1}{L}\int_{-L}^{L} (f(x) - u_0) \sin\left(\frac{(2n+1)n\pi x}{2L}\right) dx$$

And the complete solution is

$$u(x,t) = \underbrace{\sum_{n=0}^{\infty} b_n e^{-\frac{(2n+1)^2\pi^2\alpha^2}{4L^2}t} \sin\left(\frac{(2n+1)\pi x}{2L}\right)}_{\text{transient solution, decays to zero}} + \underbrace{u_0}_{\text{steady state solution}}$$

It is possible that we have a direct natural heat loss depending on the temperature a point. Then the equation itself would still be homogeneous but the boundary condition could be inhomogeneous.

**Example 17.** (Mixed Boundary condition with Heat Loss)
Consider the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} - \beta^2 u,\ 0 < x < L,\ t > 0$$

with the inhomogeneous mixed boundary condition

$$u(0,t) = u_0,\ u_x(L,t) = q_0$$

and the initial condition

$$u(x,0) = f(x)$$

First we want to find the steady state solution $w(x) = u(x,\infty)$. With $\dfrac{\partial w}{\partial t} = 0$, we have

$$\alpha^2 \frac{\partial^2 w}{\partial x^2} - \beta^2 w = 0$$

whose characteristic equation is

$$\alpha^2 r^2 - \beta^2 = 0$$

with the solution $r = \pm\dfrac{\beta}{\alpha}$. Therefore the solution is

$$w(x) = A\sinh\left(\frac{\beta}{\alpha}x\right) + B\cosh\left(\frac{\beta}{\alpha}x\right)$$

Plug in the boundary condition, we have

$$u(0,t) = w(0) = B = u_0$$

and

$$u_x(L,t) = w'(L) = \frac{\beta A}{\alpha}\cosh\left(\frac{\beta}{\alpha}L\right) + \frac{\beta B}{\alpha}\sinh\left(\frac{\beta}{\alpha}L\right) = q_1$$

We can solve to have

$$B = u_0,\ A = \frac{q_1\alpha}{\beta\cosh\left(\frac{\beta}{\alpha}L\right)} - \frac{u_0\sinh\left(\frac{\beta}{\alpha}L\right)}{\cosh\left(\frac{\beta}{\alpha}L\right)}$$

Therefore the steady state solution is

$$w(x) = \left(\frac{q_1\alpha}{\beta\cosh\left(\frac{\beta}{\alpha}L\right)} - \frac{u_0\sinh\left(\frac{\beta}{\alpha}L\right)}{\cosh\left(\frac{\beta}{\alpha}L\right)}\right)\sinh\left(\frac{\beta}{\alpha}x\right) + u_0\cosh\left(\frac{\beta}{\alpha}x\right)$$

Then we want to find the homogeneous solution $v(x,t) = u(x,t) - w(x)$. We could simplify the equation

$$\frac{\partial}{\partial t}\left(v(x,t) + w(x)\right) = \alpha^2\frac{\partial^2}{\partial x^2}\left(v(x,t) + w(x)\right) - \beta^2\left(v(x,t) + w(x)\right)$$

to be

$$\frac{\partial v}{\partial t} = \alpha^2\frac{\partial^2 v}{\partial x^2} - \beta^2 v$$

And the boundary conditions become

$$v(0,t) = u(0,t) - w(0) = u_0 - u_0 = 0$$

and

$$v_x(L,t) = u_x(L,t) - w_x(L) = q_1 - q_1 = 0$$

And the initial condition becomes

$$v(x,0) = u(x,0) - w(x) = f(x) - w(x)$$

We want to use the method of separation of variables, assuming $v(x,t) = X(x)T(t)$. We plug it into the equation

$$X(x)T'(t) = \alpha^2X''(x)T(t) - \beta^2X(x)T(t)$$

which could be simplified as

$$\frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X} - \frac{\beta^2}{\alpha^2} = \text{constant}$$

We let

$$\frac{1}{\alpha^2}\frac{T'}{T} + \frac{\beta^2}{\alpha^2} = \frac{X''}{X} = \lambda$$

We know from the previous section, $\lambda = -\mu^2 < 0$. Therefore we have two separate equation

$$T' + (\alpha^2\mu^2 + \beta^2)T = 0$$

and

$$X'' + \mu^2 X = 0,\ X(0) = 0,\ X'(L) = 0$$

Then, from the previous section, we know the solution should be

$$T_n(t) = e^{-\left(\alpha^2\left(\frac{(2n+1)\pi}{2n}\right)^2 + \beta^2\right)t}$$

and

$$X_n(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

Then we know the solution should be

$$v(x,t) = \sum_{n=0}^{\infty} b_n e^{-\left(\alpha^2\left(\frac{(2n+1)\pi}{2n}\right)^2 + \beta^2\right)t} \sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

where

$$b_n = \frac{1}{L}\int_{-L}^{L} (f(x) - w(x)) \sin\left(\frac{(2n+1)\pi x}{2L}\right)\, dx$$

We could pull out the $e^{\beta^2 t}$, representing the heat loss,

$$v(x,t) = e^{\beta^2 t}\sum_{n=0}^{\infty} b_n e^{-\left(\alpha^2\left(\frac{(2n+1)\pi}{2n}\right)^2\right)t} \sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

Therefore the complete solution should be

$$\begin{aligned} u(x,t) = &\underbrace{\left(\frac{q_1\alpha}{\beta\cosh\left(\frac{\beta}{\alpha}L\right)} - \frac{u_0\sinh\left(\frac{\beta}{\alpha}L\right)}{\cosh\left(\frac{\beta}{\alpha}L\right)}\right)\sinh\left(\frac{\beta}{\alpha}x\right) + u_0\cosh\left(\frac{\beta}{\alpha}x\right)}_{\text{steady state solution}} \\ &+ \underbrace{e^{\beta^2 t}\sum_{n=0}^{\infty} b_n e^{-\left(\alpha^2\left(\frac{(2n+1)\pi}{2n}\right)^2\right)t} \sin\left(\frac{(2n+1)\pi x}{2L}\right)}_{\text{transient solution}} \end{aligned}$$

In summary the key idea is always to find the particular solution and homogeneous solution.

$$\boxed{u(x,t) = w(x) + v(x,t)}$$

ii. **Heat Conduction with Distributed Sources/Sinks**

In this section we will encounter the heat equation with extra inhomogeneous terms, which is caused by the distributed sources or sinks. This extra term could be time-independent or time-dependent. In general, we could write the heat equation as

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} + s(x,t)$$

If the extra term is time-independent, the method to find the particular solution and homogeneous solution will still work.

**Example 18.** (Time-independent Distributed Source)
Consider the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + x,\ 0 < x < L,\ t > 0$$

with the inhomogeneous mixed boundary condition

$$u(0,t) = u_1,\ u(L,t) = u_2$$

and the initial condition

$$u(x,0) = f(x)$$

First we want to find the steady state solution $w(x) = u(x, \infty)$. With $\dfrac{\partial w}{\partial t} = 0$, we have

$$\alpha^2 \frac{\partial^2 w}{\partial x^2} + x = 0$$

which implies

$$w'(x) = -\frac{x^2}{2\alpha^2} + A$$

and then

$$w(x) = -\frac{x^3}{6\alpha^2} + Ax + B$$

Plug in the boundary condition, we have

$$u(0,t) = w(0) = B = u_1$$

and

$$u(L,t) = w(L) = -\frac{L^3}{6\alpha^2} + AL + B = u_2$$

Then we have

$$A = \frac{u_2 - u_1}{L} + \frac{L^2}{6\alpha^2},\ B = u_1$$

And the steady state solution is

$$w(x) = -\frac{x^3}{6\alpha^2} + \left(\frac{u_2 - u_1}{L} + \frac{L^2}{6\alpha^2}\right) x + u_1$$

Then we want to find the transient solution $v(x,t) = u(x,t) - w(x)$. We could simplify the solution

$$\frac{\partial}{\partial t}(v(x,t) + w(x)) = \alpha^2 \frac{\partial^2}{\partial x^2}(v(x,t) + w(x)) + x$$

to be

$$\frac{\partial v}{\partial t} = \alpha^2 \frac{\partial^2 v}{\partial x^2}$$

And the boundary condition becomes

$$v(0,t) = u(0,t) - w(0) = u_1 - u_1 = 0$$

and

$$v(L,t) = u(L,t) - w(L) = u_2 - u_2 = 0$$

And the initial condition becomes

$$v(x,0) = u(x,0) - w(x) = f(x) - w(x)$$

Therefore the solution is

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\alpha^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{1}{L}\int_{-L}^{L} (f(x) - w(x)) \sin\left(\frac{n\pi x}{L}\right)$$

Then the complete solution is

$$u(x,t) = \underbrace{-\frac{x^3}{6\alpha^2} + \left(\frac{u_2 - u_1}{L} + \frac{L^2}{6\alpha^2}\right)x + u_1}_{\text{steady state solution}} + \underbrace{\sum_{n=1}^{\infty} b_n e^{-\frac{n^2\alpha^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)}_{\text{transient solution}}$$

If the source or sink is time-dependent, we have to use the method of eigenfunction expansion.

**Example 19.** (Time-dependent Distributed Source)
Consider the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + xt,\ 0 < x < L,\ t > 0$$

with Dirichlet inhomogeneous boundary condition

$$u(0,t) = u_1,\ u(L,t) = u_2$$

and the initial condition

$$u(x,0) = f(x)$$

Since there is a $t$ term in the equation, it is actually impossible to find a steady state solution. However, we could find a particular solution to the homogeneous equation with inhomogeneous boundary condition $w(x)$ which is steady. Then we could find a particular (but also "general") solution to the inhomogeneous equation with homogeneous boundary condition $v(x,t)$. Based on the principal of superposition, we could find a complete solution $u(x,t) = w(x) + v(x,t)$.

To find the "steady state" solution, with $\dfrac{\partial w}{\partial t} = 0$ and deleting the $xt$ term, we have

$$\frac{\partial^2 w}{\partial x^2} = 0$$

which implies

$$w(x) = Ax + B$$

Plug in the boundary condition,

$$B = w(0) = u(0,t) = u_1$$

and

$$AL + B = w(L) = u(L,t) = u_2$$

we have

$$A = \frac{u_2 - u_1}{L},\ B = u_1$$

Then the particular solution is

$$w(x) = \frac{u_2 - u_1}{L} x + u_1$$

Then we want to find the solution to the inhomogeneous equation with homogeneous boundary condition $v(x,t) = u(x,t) - w(x)$. Then we could simplify the equation

$$\frac{\partial w}{\partial t} + \frac{\partial v}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + xt$$

to be

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + xt$$

And the boundary condition is

$$v(0,t) = u(0,t) - w(0) = u_1 - u_1 = 0$$

and

$$v(L,t) = u(L,t) - w(L) = u_2 - u_2 = 0$$

And the initial condition is

$$v(x,0) = u(x,0) - w(x) = f(x) - w(x)$$

Inspired by the idea of Fourier series, we could expand $s(x,t) = xt$ to the basis of sine or cosine function and solve the equation on every node. Let's expand $s(x,t) = xt$ by Fourier series. Since it is Dirichlet boundary condition, we know the eigenfunction of $X(x)$ should be $X_n(x) = \sin\left(\dfrac{n\pi x}{L}\right)$, $T_n(t) = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$. Therefore, we want to expand $s(x,t)$ in terms of

$$s(x,t) = \sum_{n=1}^{\infty} \hat{s}_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\hat{s}_n(t) = \frac{1}{L}\int_{-L}^{L} s(x,t)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

At the same time, we want to expand $v(x,t)$ in terms of

$$v(x,t) = \sum_{n=1}^{\infty} \hat{v}_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

Our target is to solve $\hat{v}_n(t)$. Then we have to find the solution on every node. Plug back to the equation:

$$\frac{\partial}{\partial t}\left(\hat{v}_n(t)\sin\left(\frac{n\pi x}{L}\right)\right) = \alpha^2\frac{\partial^2}{\partial x^2}\left(\hat{v}_n(t)\sin\left(\frac{n\pi x}{L}\right)\right) + \hat{s}_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

Simplify the equation, we get

$$\left(\hat{v}_n'(t) + \frac{n^2\pi^2\alpha^2}{L^2}\hat{v}_n(t) - \hat{s}_n(t)\right)\sin\left(\frac{n\pi x}{L}\right) = 0$$

Then we have to solve the inhomogeneous linear first order ordinary differential equation

$$\hat{v}_n'(t) + \frac{n^2\pi^2\alpha^2}{L^2}\hat{v}_n(t) = \hat{s}_n(t)$$

with the solution $\hat{v}_n(t)$. The integral factor is

$$r(t) = e^{\frac{n^2\pi^2\alpha^2}{L^2}t}$$

Then the solution should be

$$\hat{v}_n(t) = \frac{1}{r(t)}\left(\int e^{\frac{n^2\pi^2\alpha^2}{L^2}t}\hat{s}_n(t)\,dt + c_n\right) = \int_0^t e^{-\frac{n^2\pi^2\alpha^2}{L^2}(t-\tau)}\hat{s}_n(\tau)d\tau + c_n e^{\frac{-n^2\pi^2\alpha^2}{L^2}t}$$

To find the constant $c_n$, we need to plug in the initial condition,

$$f(x) - w(x) = v(x,0) = \sum_{n=1}^{\infty} c_n\sin\left(\frac{n\pi x}{L}\right)$$

This is a Fourier series, where

$$c_n = \frac{1}{L}\int_{-L}^{L}(f(x) - w(x))\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Then our complete solution would be

$$\boxed{\begin{aligned} u(x,t) = \; & \underbrace{\frac{u_2 - u_1}{L}x + u_1}_{\text{inhomogeneous boundary condition}} + \underbrace{\sum_{n=1}^{\infty}\left(\int_0^t e^{-\frac{n^2\pi^2\alpha^2}{L^2}(t-\tau)}\hat{s}_n(\tau)d\tau\right)\sin\left(\frac{n\pi x}{L}\right)}_{\text{inhomogeneous equation}} \\ & + \underbrace{\sum_{n=1}^{\infty} c_n e^{\frac{-n^2\pi^2\alpha^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)}_{\text{homogeneous solution}} \end{aligned}}$$

Back to our example, plug in $s(x,t) = xt$, we have

$$\hat{s}_n(t) = \left(\frac{2L}{n\pi}\right)(-1)^{n+1} t$$

and then

$$\int_0^t e^{-\frac{n^2\pi^2\alpha^2}{L^2}(t-\tau)} \hat{s}_n(\tau) d\,\tau = \left(\frac{2L}{n\pi}\right)(-1)^n \left(\frac{\frac{n^2\pi^2\alpha^2}{L^2}t - 1 + e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}}{\frac{n^4\pi^4\alpha^4}{L^4}}\right)$$

and

$$c_n = \frac{1}{L}\int_{-L}^{L} (f(x) - w(x)) \sin\left(\frac{n\pi x}{L}\right) dx$$

iii. **Time-dependent Boundary Condition**
To be more general, let's look at the cases where the boundary conditions are time-dependent. The basic idea is still to find the simplest particular solution to satisfy (cancel out) the boundary condition.

**Example 20.** (Dirichlet Time-dependent Boundary Condition)
Consider the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

with boundary condition

$$u(0,t) = \phi_1(t),\ u(0,t) = \phi_2(t)$$

and initial condition

$$u(x,0) = f(x)$$

Inspired by the guess in time-independent case, we let the particular solution to be

$$w(x,t) = A(t)x + B(t)$$

Plug in the boundary condition

$$w(0,t) = B(t) = \phi_1(t)$$

$$w(L,t) = A(t)L + B(t) = \phi_2(t)$$

from which we could solve $A(t) = \dfrac{\phi_2(t) - \phi_1(t)}{L}$ and $B(t) = \phi_1(t)$. Therefore the particular solution is

$$w(x,t) = \frac{\phi_2(t) - \phi_1(t)}{L}x + \phi_1(t)$$

Then we want to find the homogeneous solution $v(x,t) = u(x,t) - w(x,t)$. We plug it into the PDE, to get

$$\frac{\partial w}{\partial t} + \frac{\partial v}{\partial t} = \alpha^2 \frac{\partial^2 w}{\partial x^2} + \alpha^2 \frac{\partial^2 v}{\partial x^2}$$

With $\dfrac{\partial^2 w}{\partial x^2} = 0$ and $\dfrac{\partial w}{\partial t} = \dfrac{\phi_2'(t) - \phi_1'(t)}{L}x + \phi_1'(t)$, we could rearrange the equation to be

$$\frac{\partial v}{\partial t} = \alpha^2 \frac{\partial^2 w}{\partial x^2} - \left\{ \frac{\phi_2'(t) - \phi_1'(t)}{L}x + \phi_1'(t) \right\}$$

with boundary condition

$$v(0,t) = u(0,t) - w(0,t) = \phi_1(t) - \phi_1(t) = 0$$

$$v(L,t) = u(L,t) - w(L,t) = \phi_2(t) - \phi_2(t) = 0$$

and initial condition

$$v(x,0) = u(x,0) - w(x,0) = f(x,0) - \frac{\phi_2(0) - \phi_1(0)}{L}x - \phi_1(0)$$

Then we could use the eigenfunction expansion with sine functions basis to solve it.

Similarly, we could make the following guesses with the following boundary conditions.

| Type | Boundary Conditions | Guess |
|---|---|---|
| Neumann | $u_x(0,t) = \phi_1(t),\ u_x(L,t) = \phi_2(t)$ | $w(x,t) = \dfrac{\phi_2(t) - \phi_1(t)}{2L}x^2 + \phi_1(t)x$ |
| Mixed I | $u_x(0,t) = \phi_1(t),\ u(L,t) = \phi_2(t)$ | $w(x,t) = \phi_2(t)x + \phi_1(t)$ |
| Mixed II | $u_x(0,t) = \phi_1(t),\ u(L,t) = \phi_2(t)$ | $w(x,t) = \phi_1(t)x + (\phi_2(t) - \phi_1(t)L)$ |

And with the same method we could have the equation of the homogeneous solution $v(x,t) = u(x,t) - w(x,t)$.

2. **Wave Equation**

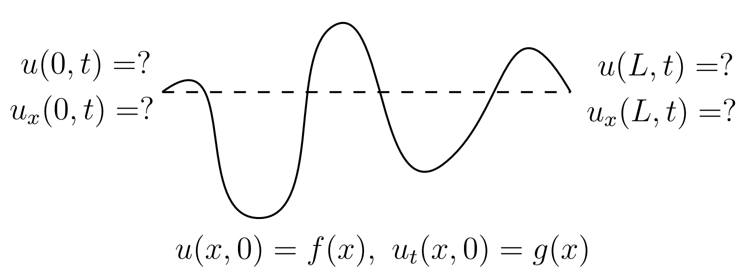
In general, a wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with two boundary conditions and two initial conditions

$$u(x,0) = f(x),\ u_t(x,0) = g(x)$$

For example, consider a string vibrating



(a) **Galilean Transformation and D'Alembert's Solution**
Recall in the introduction of the PDE, we guess a solution

$$u(x,t) = e^{ikx+\sigma t}$$

with the dispersion relation

$$\sigma^2 = -c^2k^2$$

Then the solution is

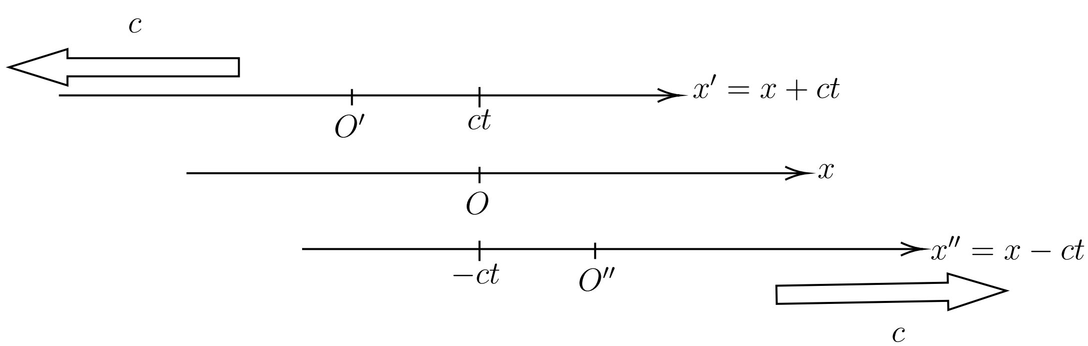
$$u(x,t) = e^{ik(x\pm ct)}$$

To generalize the solution, we let $u_1(x,t) = L(x+ct)$ and $u_2(x,t) = R(x-ct)$. Plug them in the equation

$$c \cdot c \cdot L'' = c^2L''$$

$$(-c)\cdot(-c)\cdot R'' = c^2R''$$

which both are solutions. Consider them in Galilean Transformation, $L(x+ct)$ represent the solution of a coordinate system moving left with speed $c$ ($x' = x+ct$) and $R(x-ct)$ represent the solution of a coordinate system moving right with speed $c$ ($x'' = x-ct$).




Since the wave equation is linear, and $L$ and $R$ are arbitrary, we know the solution could be the superposition

$$u(x,t) = R(x-ct) + L(x+ct)$$

To find out $R$ and $L$, we are going to apply the initial condition with no boundary condition. In other solution works on $(-\infty,\infty)$. We have

$$u(x,0) = R(x) + L(x) = f(x) \tag{*}$$

and

$$u_t(x,0) = -cR'(x) + cL'(x) = g(x)$$

which implies

$$-cR(x) + cL(x) = \int_0^x g(s)\,ds + A \tag{**}$$

With $(*)$ and $(**)$, we have

$$R(x) = \frac{1}{2}f(x) - \frac{1}{2c}\left(\int_0^x g(s)\,ds + A\right)$$

and

$$L(x) = \frac{1}{2}f(x) + \frac{1}{2c}\left(\int_0^x g(s)\,ds + A\right)$$

Therefore, the solution is

$$\begin{aligned} u(x,t) &= L(x+ct) + R(x-ct) \\ &= \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\left(\int_0^{x+ct} g(s)\,ds + A - \int_0^{x-ct} g(s)\,ds - A\right) \\ &= \underbrace{\frac{1}{2}\left(f(x+ct) + f(x-ct)\right)}_{\text{left and right wave from initial displacement}} + \underbrace{\frac{1}{2c}\left(\int_{x-ct}^{x+ct} g(s)\,ds\right)}_{\text{determined by initial velocity}} \end{aligned}$$

This is solution is D'Alembert's solution in the domain $(-\infty, \infty)$ - no boundary.

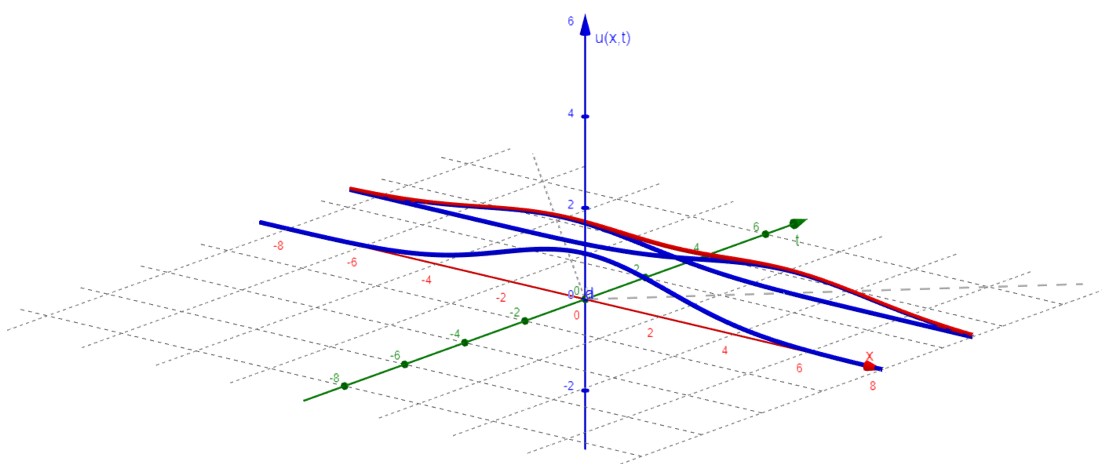
(b) **Space-time Interpretation of D'Alembert's Solution**
To observe how the initial condition determine the solution, we simplify it that $g(x) = 0$. Therefore the D'Alembert's solution becomes

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$$

For example we have a impulse $u(x,0) = f(x) = e^{-0.1\cdot x^2}$ at $t = 0$ and $c = 1$. Then the solution at $t = 3$ would be

$$u(x,3) = \frac{1}{2}\left(e^{-0.1\cdot(x-3)^2} + e^{-0.1\cdot(x+3)^2}\right)$$

As shown in the graph below,

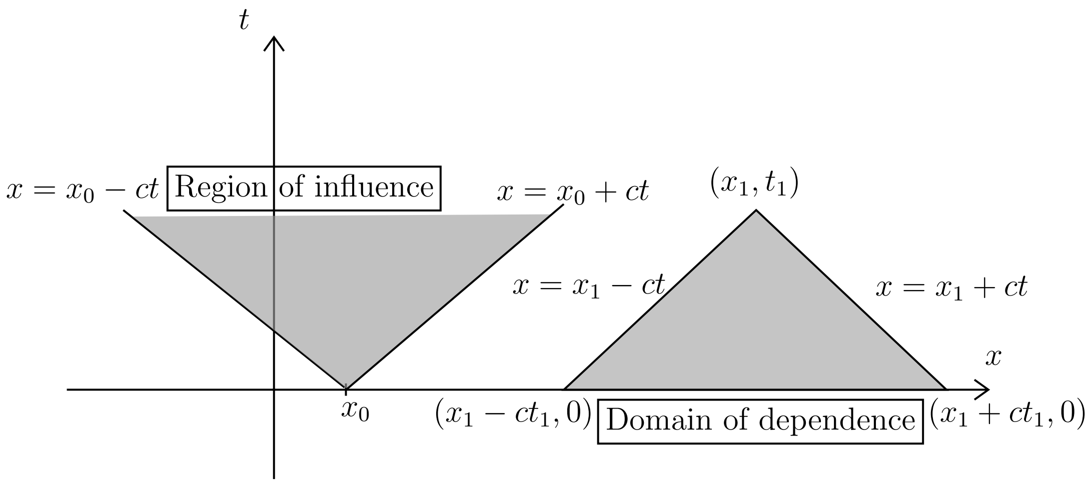


the wave splits into two waves travelling to right and left with speed and the solution is the superposition of waves. We could see the center of the impulse travels along $x = ct$ and $x = -ct$. Therefore we call the region between $x = ct$ and $x = -ct$ $(t \geq 0)$ the *region of influence* by $(0, 0)$. In the other way, we could look at point $(0, 3)$ where $u(0, 3) = \frac{1}{2}(f(-3) + f(3))$ depending on the two points $(-ct, 0)$ and $(ct, 0)$. Therefore we call the region between $x = 3-ct$ and $x = 3+ct$ $(t \geq 0)$ the *domain of dependence*.

This would also work for the case $g(x) \neq 0$. Then we give the general definition.

**Definition 2.** (Region of influence and Domain of dependence)

- For a point $(x_0, 0)$ as an initial condition point, the lines $x+ct = x_0$ and $x-ct = x_0$ bound the *region of influence* of the function values at the initial point $(x_0, 0)$.
- For a point $(x_1, t_1)$ in the future of the space-time diagram, the lines $x = x_1-ct_1$ and $x = x_1 + ct_1$ that pass through the point $(x_1, t_1)$ bound the *domain of dependence* of $(x_1, t_1)$.




**Example 21.**

(c) **Solution by Separation of Variables**

Recall the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0$$

with two boundary conditions and two initial conditions

$$u(x, 0) = f(x),\ u_t(x, 0) = g(x)$$

We want to apply the separation of variables

$$u(x, y) = X(x) \cdot T(t)$$

**Example 22.** (Wave Equation with Dirichlet Condition)
Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0$$

with two boundary conditions

$$u(0,t) = u(L,t) = 0$$

and two initial conditions

$$u(x,0) = f(x),\ u_t(x,0) = g(x)$$

Plug in $u(x,t) = X(x)T(t)$, we have

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\mu^2$$

Based on the boundary condition, we know the solution of $X$.

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right),\ \lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2$$

Corresponding we can find out the solution of $T$.

$$T_n(t) = A_n \cos\left(c\mu_n t\right) + B_n \sin\left(c\mu_n t\right) = A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)$$

Then the complete solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

To find out $A_n$ and $B_n$ we need to plug in the initial condition. We have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right) = g(x)$$

With Fourier transformation, we have

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

and

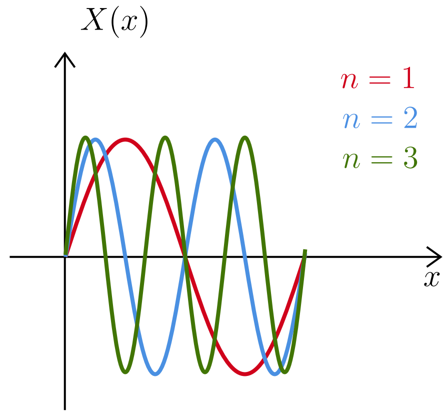
$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

In this example, we could treat each $X_n(x)$, $T_n(t)$ as a mode, with the wavelength to be

$$\lambda_n = \frac{2L}{n}$$

and

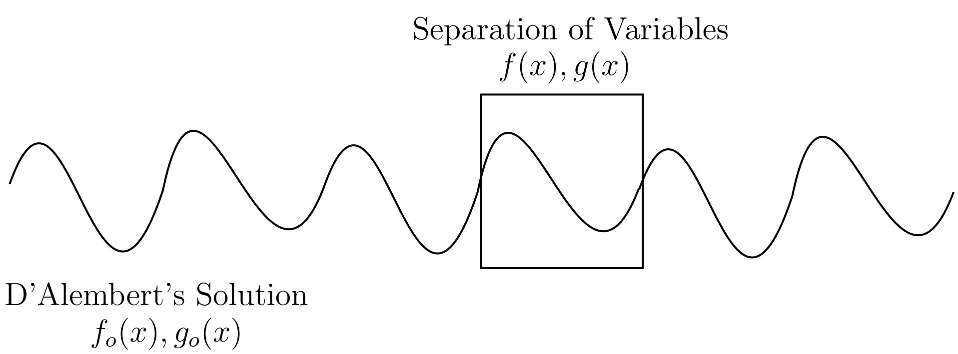
$$T_n = \frac{2L}{nc},\ f_n = \frac{1}{T_n} = \frac{nc}{2L}$$




(d) **Comparison between D'Alembert's Solution and the Solution by Separation of Variables**
With trigonometry identity, we have the solution by separation of variables to be

$$
\begin{aligned}
u(x,t) = \sum_{n=1}^{\infty} \frac{1}{2} \Bigg( & A_n \sin\left(\frac{n\pi(x+ct)}{L}\right) + A_n \sin\left(\frac{n\pi(x-ct)}{L}\right) \\
& B_n \cos\left(\frac{n\pi(x-ct)}{L}\right) - B_n \cos\left(\frac{n\pi(x+ct)}{L}\right)\Bigg) \\
= \frac{1}{2} \Bigg( & \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi(x+ct)}{L}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi(x-ct)}{L}\right) \\
& \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi(x-ct)}{L}\right) - \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi(x+ct)}{L}\right)\Bigg) \\
= \frac{1}{2} & \left(f_o(x+ct) + f_o(x-ct)\right) + \frac{1}{2c}\left(\int_{x-ct}^{x+ct} g_o(s)\,ds\right)
\end{aligned}
$$

where $f_o(x)$, $g_o(x)$ are the odd periodic extension of $f(x)$ and $g(x)$. In conclusion, if we know a Dirichlet boundary condition, we have the D'Alembert's Solution in which the initial displacement function is given by the odd periodic extension $f_o$ of the initial displacement of the string, and the initial velocity function is given by the odd periodic extension $g_o$ of the initial velocity of the string.



3. **Laplace's Equation**

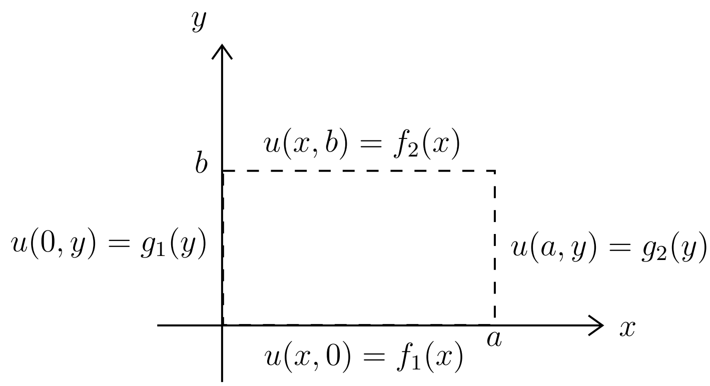
(a) **Rectangular Domain**

A Laplace's equation is

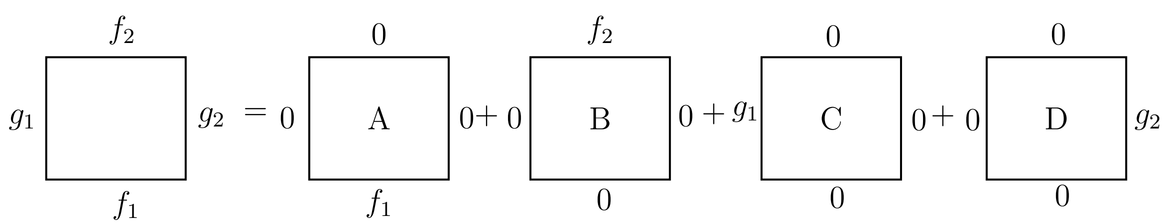
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b$$

with four boundary conditions

$$u(x,0) = f_1(x),\ u(x,b) = f_2(x)$$
$$u(0,y) = g_1(y),\ u(a,y) = g_2(y)$$




The basic idea is to separate the boundary condition




**Example 23.** (Dirichlet Boundary Condition A)
Consider the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b$$

with four boundary conditions

$$u(x,0) = f_1(x),\ u(x,b) = 0$$
$$u(0,y) = 0,\ u(a,y) = 0$$

We want to use the separation of variable $u(x,y) = X(x)Y(y)$ implying

$$-\frac{Y''}{Y} = \underbrace{\frac{X''}{X} = \lambda}_{\text{homo BC}} = -\mu^2$$

Then for $X(x)$, with $X(0) = X(a) = 0$, we know the solution is

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right),\ \lambda_n = -\left(\frac{n\pi}{a}\right)^2$$

Then for $Y(y)$, we have the equation

$$Y'' - \left(\frac{n\pi}{a}\right)^2 Y = 0$$

with the solution

$$Y_n(y) = A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right)$$

Therefore the complete solution is

$$u(x,y) = \sum_{n=1}^{\infty} \left(A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right)\right) \sin\left(\frac{n\pi x}{a}\right)$$

Then we want to apply the boundary condition,

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi x}{a}\right) = f_1(x)$$

Then with Fourier series expansion, we have

$$A_n = \frac{2}{a}\int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right)\, dx$$

With another boundary condition

$$u(x,b) = \sum_{n=1}^{\infty} \left(A_n \cosh\left(\frac{n\pi b}{a}\right) + B_n \sinh\left(\frac{n\pi b}{a}\right)\right) \sin\left(\frac{n\pi x}{a}\right) = 0$$

which implies

$$B_n = -\frac{A_n}{\tanh\left(\frac{n\pi b}{a}\right)}$$

Then the solution is

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \frac{A_n}{\sinh\left(\frac{n\pi b}{a}\right)} \left(\sinh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) - \cosh\left(\frac{n\pi b}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\right)\sin\left(\frac{n\pi x}{a}\right) \\
&= \sum_{n=1}^{\infty} \frac{A_n}{\sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi(b-y)}{a}\right)\sin\left(\frac{n\pi x}{a}\right)
\end{aligned}$$

**Example 24.** (Dirichlet Boundary Condition B)
Consider the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b$$

with four boundary conditions

$$u(x,0) = 0,\ u(x,b) = 0$$

$$u(0,y) = g_1(y),\ u(a,y) = 0$$

We want to use the separation of variable $u(x,y) = X(x)Y(y)$ implying

$$-\frac{X''}{X} = \underbrace{\frac{Y''}{Y}}_{\text{homo BC}} = \lambda = -\mu^2$$

Then for $Y(y)$, with $Y(0) = Y(b) = 0$, we know the solution is

$$Y_n(y) = \sin\left(\frac{n\pi y}{b}\right),\ \lambda_n = -\left(\frac{n\pi}{b}\right)^2$$

Then for $X(x)$, we have the equation

$$X'' - \left(\frac{n\pi}{b}\right)^2 X = 0$$

with the solution

$$X_n(x) = A_n \cosh\left(\frac{n\pi x}{b}\right) + B_n \sinh\left(\frac{n\pi x}{b}\right)$$

Therefore the complete solution is

$$u(x,y) = \sum_{n=1}^{\infty}\left(A_n \cosh\left(\frac{n\pi x}{b}\right) + B_n \sinh\left(\frac{n\pi x}{b}\right)\right)\sin\left(\frac{n\pi y}{b}\right)$$

Then we want to apply the boundary condition,

$$u(0,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi y}{b}\right) = g_1(y)$$

Then with Fourier series expansion, we have

$$A_n = \frac{2}{b}\int_0^b g_1(y)\sin\left(\frac{n\pi y}{b}\right)\,dy$$

With another boundary condition

$$u(a,y) = \sum_{n=1}^{\infty}\left(A_n \cosh\left(\frac{n\pi a}{b}\right) + B_n \sinh\left(\frac{n\pi a}{b}\right)\right)\sin\left(\frac{n\pi y}{b}\right) = 0$$

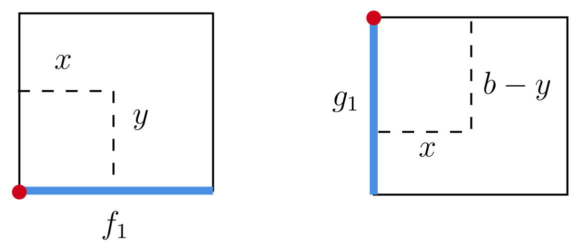
which implies

$$B_n = -\frac{A_n}{\tanh\left(\frac{n\pi a}{b}\right)}$$

Then the solution is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \frac{A_n}{\sinh\left(\frac{n\pi a}{b}\right)} \left( \sinh\left(\frac{n\pi a}{b}\right) \cosh\left(\frac{n\pi x}{b}\right) - \cosh\left(\frac{n\pi a}{b}\right) \sinh\left(\frac{n\pi x}{b}\right) \right) \sin\left(\frac{n\pi y}{b}\right) \\
&= \sum_{n=1}^{\infty} \frac{A_n}{\sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi (a-x)}{b}\right) \sin\left(\frac{n\pi y}{b}\right)
\end{aligned}
$$

**Note:** Comparing these two examples, we can find the geometric symmetry between them.




$$
\begin{aligned}
x &\leftrightarrow b-y \\
y &\leftrightarrow x \\
a &\leftrightarrow b
\end{aligned}
$$

If the boundary condition becomes to be Neumann, the case is still very similar.

**Example 25.** (Neumann Boundary Condition)
Consider the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b$$

with four boundary conditions

$$u_y(x,0) = 0,\ u_y(x,b) = 0$$

$$u_x(0,y) = g_1(y),\ u_x(a,y) = 0$$

We want to use the separation of variable $u(x,y) = X(x)Y(y)$ implying

$$-\frac{X''}{X} = \underbrace{\frac{Y''}{Y}}_{\text{homo BC}} = \lambda = -\mu^2$$

Then for $Y(y)$, with $Y'(0) = Y'(b) = 0$, we know the solution is

$$Y_n(y) = \left\{1,\ \cos\left(\frac{n\pi y}{b}\right)\right\},\ \lambda = -\left(\frac{n\pi}{b}\right)^2$$

Then for $X(x)$, we have the equation to be

$$X'' - \left(\frac{n\pi}{b}\right)^2 X = 0$$

with the solution

$$X_n(x) = A_n \cosh\left(\frac{n\pi x}{b}\right) + B_n \sinh\left(\frac{n\pi x}{b}\right),\ n > 0$$

and

$$X_n(x) = Cx + D,\ n = 0$$

With the boundary condition $X'(a) = 0$, we know

$$B_n = -A_n \tanh\left(\frac{n\pi a}{b}\right)$$

and

$$C = 0$$

Then we know the solution of $X(x)$ could be simplified to be

$$\begin{aligned} X_n(x) &= \frac{A_n}{\cosh\left(\frac{n\pi a}{b}\right)}\left(\cosh\left(\frac{n\pi a}{b}\right)\cosh\left(\frac{n\pi x}{b}\right) - \sinh\left(\frac{n\pi a}{b}\right)\sinh\left(\frac{n\pi x}{b}\right)\right) \\ &= \frac{A_n}{\cosh\left(\frac{n\pi a}{b}\right)}\cosh\left(\frac{n\pi(x-a)}{b}\right) \end{aligned}$$

Then the complete solution is

$$u(x,y) = D + \sum_{n=1}^{\infty} \frac{A_n}{\cosh\left(\frac{n\pi a}{b}\right)}\cosh\left(\frac{n\pi(x-a)}{b}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Then we could apply the last boundary condition

$$u_x(0,y) = \sum_{n=1}^{\infty} -A_n\left(\frac{n\pi}{b}\right)\tanh\left(\frac{n\pi a}{b}\right)\cos\left(\frac{n\pi y}{b}\right) = g_1(y)$$

With Fourier series expansion, we have

$$A_n = -\frac{2}{n\pi\tanh\left(\frac{n\pi a}{b}\right)}\int_0^b g_1(y)\cos\left(\frac{n\pi y}{b}\right)\,dy$$

To be consistent, the constant term

$$A_0 = -\frac{2}{n\pi\tanh\left(\frac{n\pi a}{b}\right)}\int_0^b g_1(y)\,dy = 0 \tag{*}$$

which is also

$$\int_0^b g_1(y)\,dy = 0$$

We could interpret (*) physically. Recall a Laplace equation could be the steady state solution function of a 2D heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

If there is no net flow

$$\int_0^b g_1(y)\,dy = 0$$

Then it is possible to have a steady state solution. But look back to the solution above, we still don't know the constant $D$. Then $u(x,y)$ is said to be known up to an arbitrary constant.
To find out the constant, we should still think about it physically. Since there is no net flow, the total amount of heat would be unchanged, then the constant could be the average of heat at any moment, where we use the heat distribution at $t=0$,

$$D = \frac{1}{\text{Area}} \int u_0(x,y)\, dxdy = \frac{1}{ab} \int u_0(x,y)\, dxdy$$

Sometimes we would find that the boundary is open, then it becomes a bit similar to the heat equation problem.
**Example 26.** (Semi-infinite strip problem)
Consider the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b$$

with four boundary conditions

$$u_y(x,0) = f(x),\ u_y(x,\infty) = \frac{B-A}{a}x + A$$

$$u_x(0,y) = A,\ u_x(a,y) = B$$

First we need to find the $w(x)$ as a special solution to the equation. With $w_{xx} = 0$, we know $w(x) = kx + m$. Plug in the boundary condition, solving $k = \dfrac{B-A}{a}$ and $m = A$. Then we have

$$w(x) = \frac{B-A}{a}x + A$$

Then we want to find out $v(x,y) = u(x,y) + w(x)$. Check the boundary condition, we have

$$v(x,0) = u(x,0) - w(x) = f(x) - w(x)$$

and

$$v(x,\infty) = u(x,\infty) - w(x) = \frac{B-A}{a}x + A - \left(\frac{B-A}{a}x + A\right) = 0$$

and

$$v(0,y) = u(0,y) - w(0) = A - A = 0$$

and

$$v(a,y) = u(a,y) - w(a) = B - B = 0$$

With separation of variables, $v(x,y) = X(x)Y(y)$, we have

$$-\frac{Y''}{Y} = \underbrace{\frac{X''}{X} = \lambda} = -\mu^2$$

With $X(0) = X(a) = 0$, we know

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \lambda_n = -\left(\frac{n\pi}{a}\right)^2$$

Then for $Y(y)$, we know

$$Y'' - \left(\frac{n\pi}{a}\right)^2 Y = 0$$

with the solution

$$Y_n(y) = C_n e^{\frac{n\pi}{a}y} + D_n e^{-\frac{n\pi}{a}y}$$

Plug in the boundary condition, we have

$$Y(\infty) = C_n \times \infty = 0$$

Then we need $C_n = 0$. Therefore the complete solution is

$$v(x, y) = \sum_{n=1}^{\infty} D_n e^{-\frac{n\pi}{a}y} \sin\left(\frac{n\pi x}{a}\right)$$

Plug in the last boundary condition

$$u(x, 0) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{a}\right) = f(x) - w(x)$$

With Fourier series expansion, we know

$$D_n = \frac{2}{a}\int_0^a (f(x) - w(x))\, dx$$

Then the solution is

$$u(x, y) = w(x) + v(x, y) = \frac{B - A}{a}x + A + \sum_{n=1}^{\infty} D_n e^{-\frac{n\pi}{a}y} \sin\left(\frac{n\pi x}{a}\right)$$

(b) **Circular domains**

The key point of Laplace equation is

$$\triangle u = 0$$

which could be written in different coordinates. In this section we are going to discuss the Laplace equation in polar coordinate which would be very powerful for the circular domain situation. According to the knowledge in curvilinear coordinates

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Therefore the Laplace equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Then we use the separation of variables, $u(r,\theta) = R(r)\Theta(\theta)$. We have

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

which could be simplified to be

$$-\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta} = \lambda = -\mu^2$$

Then we have two type of boundary value problems:

$$\Theta'' - \lambda\Theta = 0$$

and

$$r^2R'' + rR' + \lambda R = 0$$

We have been very familiar with the BVP for $\Theta(\theta)$. For the BVP for $R(r)$, if $\mu = 0$, the indicial equation is

$$\gamma(\gamma - 1) + \gamma = 0$$

with the solution

$$\gamma_{1,2} = 0$$

and we know

$$R(r) = A + B\ln r$$

If $\mu \neq 0$, the indicial equation is

$$\gamma(\gamma - 1) + \gamma - \mu^2 = 0$$

with the solution

$$\gamma_{1,2} = \pm\mu$$

and we know

$$R(r) = Ar^{\mu} + Br^{-\mu}$$

Then the general solution is

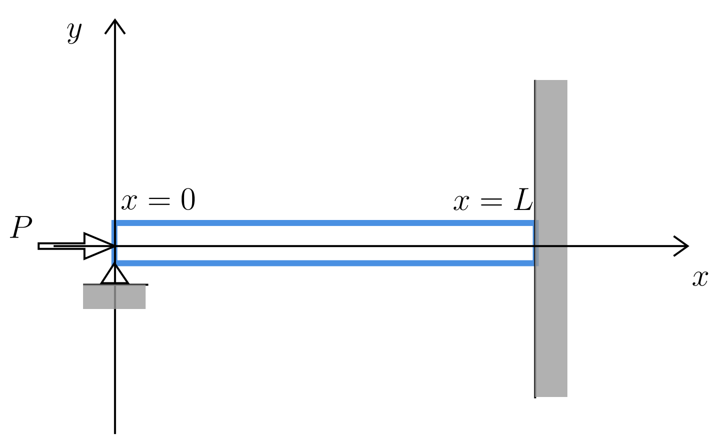
$$\begin{aligned} u(r,\theta) =& A_0 + \alpha_0 \ln r + \sum_{n=1}^{\infty} A_n r^{\sqrt{-\lambda_n}} + \alpha_n r^{-\sqrt{-\lambda_n}} \cos\sqrt{-\lambda_n}\theta + \sum_{n=1}^{\infty} B_n r^{\sqrt{-\lambda_n}} \\ &+ \beta_n r^{-\sqrt{-\lambda_n}} \sin\sqrt{-\lambda_n}\theta \end{aligned}$$

## Boundary Value Problems and Sturm-Louiville Theory

1. **Motivation**
From last section, we could see boundary value problem is everywhere. In summary, for $X''(x) + \lambda X(x) = 0$, $0 < x < L$, we have
   - Dirichlet: $X(0) = 0 = X(L)$. Then
$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \lambda_n = -\left(\frac{n\pi x}{L}\right)^2, n \in \mathbb{Z}^+$$
   - Neumann: $X'(0) = 0 = X'(L)$. Then
$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \lambda_n = -\left(\frac{n\pi x}{L}\right)^2, n \in \mathbb{Z}_0^+$$
   - Mixed I: $X'(0) = 0 = X(L)$. Then
$$X_n(x) = \cos\left(\frac{(2n+1)\pi x}{2L}\right), \lambda_n = -\left(\frac{(2n+1)\pi x}{2L}\right)^2, n \in \mathbb{Z}_0^+$$
   - Mixed II: $X(0) = 0 = X'(L)$. Then
$$X_n(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right), \lambda_n = -\left(\frac{(2n+1)\pi x}{2L}\right)^2, n \in \mathbb{Z}_0^+$$
   - Periodic: $X(-L) = X(L)$, $X'(-L) = X'(L)$. Then
$$X_n(x) = \left\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}, \lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, n \in \mathbb{Z}_0^+$$

   From many aspects, BVP is very different from IVP. But just as IVP, BVP has a wide application in practice.

   **Example 27.** (Buckling of an Elastic Column)
An investigation of the buckling of a uniform elastic column of length $L$ by an axial load $P$ leads to the differential equation
$$y^{(4)} + \lambda y'' = 0, 0 < x < L$$



The parameter $\lambda$ is equal to $P/EI$, where $E$ is Young's modulus and $I$ is the moment of inertia of the cross section about an axis through the centroid perpendicular to the $xy$-plane. The boundary conditions at $x = 0$ and $x = L$ depend on how the ends of the column are supported. Typical boundary conditions are

$$y = y' = 0, \text{ clamped end}$$

$$y = y'' = 0, \text{ simply supported (hinged) end}$$

For example, it could be

$$y(0) = y'(0) = 0,\ y(L) = y'(L) = 0$$

2. **Sturm-Louiville Eigenvalue Problems**

To study more general boundary value problem, we define a certain class of boundary value problem as a generalization (interface).

**Definition 3.** (Sturm-Louiville Eigenvalue Problem) Sturm-Louiville eigenvalue problem consists a differential equation

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0,\ 0 < x < \ell \tag{1}$$

where $p(x), p'(x), q(x), r(x)$ are continuous on $[0, \ell]$ and $p(x) \geq 0$, $r(x) \geq 0$ on $[0, \ell]$, and the boundary condition

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0,\ \beta_1 y(L) + \beta_2 y'(L) = 0 \tag{2}$$

.

There are few points we could notice:

(a) Plugging in different value of $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, we could obtain Dirichlet, Neumann, Mixed boundary condition. However, periodic boundary condition is not in the scope of Sturm-Louiville eigenvalue problem because the boundary condition is not separable.

(b) If $p(x) > 0$, $r(x) > 0, \ell < \infty$, then the S-L problem is *regular*. Otherwise, (to say, $\exists x \in [0, \ell], p(x) >$ or $r(x) > 0$ or $\ell = \infty$), the S-L problem is *singular*.

(c) A second order linear differential equation (eigenvalue problem form)

$$-P(x)y'' - Q(x)y' + R(x)y = \lambda y$$

could be reduced to a S-L form with a factor $\mu(x)$. With the factor we have

$$-P(x)\mu(x)y'' - Q(x)\mu(x)y' + R(x)\mu(x)y = \lambda\mu(x)y$$

By expanding the S-L equation, we have

$$-p(x)y'' - p'(x)y' + q(x)y = \lambda r(x)y$$

Then we have $P\mu = p$ and $Q\mu = p'$ with $q = R\mu$ and $r = \mu$. Therefore we know

$$P'\mu + P\mu' = Q\mu$$

in the form of

$$\frac{d\mu}{dx} = \frac{Q - P'}{P}\mu$$

Therefore we know the factor is

$$\mu(x) = e^{\int \frac{Q-P'}{P}\,dx} = e^{\int \frac{Q}{P}\,dx} e^{\int -\frac{P'}{P}\,dx} = \frac{e^{\int \frac{Q}{P}\,dx}}{P}$$

3. **Properties of Sturm-Louiville Eigenvalue Problems**

   (a) **Eigenvalues**

      i. There infinite eigenvalues of S-L problem and all eigenvalues are real.

      ii. If $\dfrac{\alpha_1}{\alpha_2} < 0$, $\dfrac{\beta_1}{\beta_2} > 0$ and $q(x) > 0$, then all eigenvalues are positive.

   (b) **Eigenfunctions**

      i. For each eigenvalues $\lambda_i$ there exists a real unique eigenfunction $\phi_i(x)$ up to a multiplicative constant and $\phi_i(x)$ has exactly $i - 1$ roots on $(0, \ell)$.

      ii. The eigenfunctions are orthogonal and can be normalized

$$\int_0^\ell r(x)\phi_n(x)\phi_m(x)\,dx = \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

   (c) **Eigenfunction Expansion**

   If $f(x)$ is piecewise smooth then $f(x)$ could be expanded as

$$\sum_{n=1}^{\infty} c_n\phi_n(x) = \frac{f(x+) + f(x-)}{2}$$

   where

$$c_n = \frac{\displaystyle\int_0^\ell r(x)f(x)\phi_n(x)\,dx}{\displaystyle\int_0^\ell r(x)\phi_n^2(x)\,dx}$$

   and $\phi_i(x)$ are not necessary to be normalized. To investigate it, we could have

$$\begin{aligned}\int_0^\ell r(x)f(x)\phi_m(x)\,dx &= \int_0^\ell r(x)\sum_{n=1}^{\infty} c_n\phi_n(x)\phi_m(x)\,dx \\ &= \sum_{n=1}^{\infty}\int_0^\ell r(x)c_n\phi_m(x)\phi_n(x)\,dx \\ &= c_m\int_0^\ell r(x)\phi_m^2(x)\,dx\end{aligned}$$

Let's look at an example with nonhomogeneous boundary condition.

**Example 28.** (Robin Boundary Condition)
Consider the ODE

$$X'' + \lambda X = 0$$

where $\lambda = \mu^2$ and with the Robin boundary condition

$$X'(0) = h_1 X(0),\ X'(\ell) = -h_2 X(\ell)$$

where $h_1, h_2 > 0$. With indicial equation, we find out that

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

Plug it back to the equation, we have

$$B\mu = h_1 A$$

and

$$-A\mu\sin(\mu\ell) + B\mu\cos(\mu\ell) = -h_2 A\cos(\mu ell) - h_2 B\sin(\mu ell)$$

Then we could solve

$$\frac{h_1}{\mu} = \frac{\mu\sin(\mu\ell) - h_2\cos(\mu\ell)}{\mu\cos(\mu\ell) + h_2\sin(\mu\ell)}$$

which can be simplified as

$$\tan(\mu\ell) = \frac{\mu(h_2 + h_1)}{\mu^2 - h_2 h_1}$$

And we know

$$X_n(x) = \frac{\mu_n}{h_1}\cos\mu_n x + \sin\mu_n x$$

Then if

- $h_1, h_2 \neq 0$: $\mu_n \approx \dfrac{n\pi}{\ell}$
- $h_1 \neq 0$ and $h_2 = 0$: $\tan(\mu_n \ell) = \dfrac{1}{\mu_n}$
- $h_1 \to \infty$ and $h_2 \neq 0$: $X_n(x) = \sin(\mu x)$, $\mu_n \approx \dfrac{2n+1}{2}\dfrac{\pi}{\ell}$

Let's look at another example related to Cauchy-Euler equation.

**Example 29.** (Cauchy-Euler Equation Eigenvalue Problem)
Consider the equation

$$-(x^{-2}y')' = \lambda x^{-4} y,\ 1 < x < 2,\ \lambda > \frac{9}{4}$$

with boundary condition

$$y(1) = y(2) = 0$$

We expand the equation, get

$$x^2y'' - 2xy' + \lambda y = 0$$

We guess $y = x^r$ to get the indicial equation $r(r-1) - 2r + \lambda = 0$ with the solution

$$r = \frac{3}{2} + i\sqrt{\lambda - \frac{9}{4}}$$

Then the solution should be

$$y_n(x) = x^{\frac{3}{2}}\left(A_n \cos\left(\sqrt{\lambda - \frac{9}{4}}\ln x\right) + B_n\left(\sqrt{\lambda - \frac{9}{4}}\ln x\right)\right)$$

Plug in the boundary condition, we have

$$y_n(1) = A_n = 0$$

and

$$y_n(2) = 2\sqrt{2}B_n \sin\left(\sqrt{\lambda - \frac{9}{4}}\ln 2\right) = 0$$

For the non-trivial solution, we need

$$\sqrt{\lambda - \frac{9}{4}}\ln 2 = n\pi$$

which gives the eigenvalue

$$\lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 + \frac{9}{4}$$

Therefore the eigenfunction is

$$y_n(x) = x^{\frac{3}{2}}\sin\left(\sqrt{\lambda_n - \frac{9}{4}}\ln x\right)$$

With $r(x) = x^{-4}$, we can expand a function $f(x)$ as

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

where

$$c_n = \frac{\displaystyle\int_1^2 x^{-4}f(x)x^{\frac{3}{2}}\sin\left(\sqrt{\lambda_n - \frac{9}{4}}\ln x\right)\,dx}{\displaystyle\int_1^2 x^{-1}\sin^2\left(\sqrt{\lambda_n - \frac{9}{4}}\ln x\right)\,dx}$$