Appendix A: Proof of Proposition 1

We first prove that if $\tau_j(\xi) < +\infty$, then (9) holds and the inequality is strict only if $c(\tau_j(\xi);\xi) \in \mathcal{K}$. The infimum in Equation (8) is always attained by the assumption that $L(c(\tau;\xi), j, \tau)$ is right-continuous in τ. So when a patient's waiting time reaches $\tau_j(\xi)$, his score must be either equal to or greater than S_J. We next prove that the latter case could happen only when the patient's health score $c(\tau_j(\xi);\xi) \in \mathcal{K}$. We prove by contradiction. Suppose $L(c(\tau_j(\xi);\xi), j, \tau_j(\xi)) > S_j$. If $c(\tau_j(\xi);\xi) \notin \mathcal{K}$, then $L(c(\tau;\xi)), j, \tau)$ continuously and strictly increases with τ in a neighborhood of $\tau_j(\xi)$. Thus, we can always find a time t which is slightly smaller than $\tau_j(\xi)$, but with $L(c(t;\xi), j, t)$ still having a larger value than S_j. That, however, contradicts with $\tau_j(\xi)$ being the first time that the patient's score has reached or exceeded S_j.

We next construct the strategy $a(\cdot,\cdot|\xi,\eta)$ of a patient with parameters (ξ,η) at a Nash equilibrium, which also proves the existence of the Nash equilibrium. For $j = 1, 2, \ldots, J$, define

$$S_j = \inf\{s \mid \iiint \chi\left(L(c(\tau;\xi), j, \tau) > s,\, a(\tau, j|\xi,\eta) = \mathrm{A}\right) \pi(c(\tau;\xi), \tau, \eta)\, d\xi d\tau d\eta = 0\}, \tag{38}$$

where $\chi(\cdot)$ represents a characteristic function. Intuitively, S_j represents the minimum threshold such that no patients with score greater than S_j are willing to accept kidneys of type j. We prove the Proposition by induction on the type of kidney offer $j = J, J-1, \ldots, 1$.

If $j = J$ (the best kidney type), then by Lemma 1, every patient's strictly dominating strategy is to accept a type-J kidney immediately when being offered. Therefore, the equilibrium strategy must have $a(\tau, J|\xi,\eta) = \mathrm{A}$ for all ξ, η, τ. Once we have determined $a(\cdot, J|\cdot,\cdot)$, S_J can be uniquely determined by (38) for any given population distribution $\Pi = (\pi(\cdot,\cdot,\cdot))$. We next prove that if $\tau_J(\xi)$ is calculated from S_J from Equation (38) and is finite, then $\tau_J(\xi)$ must be the first time that the score of patient ξ is equal to or larger than S_J.

When a patient has waited less than $\tau_J(\xi)$ time units, then his score s must be strictly smaller than S_J by the definition of $\tau_J(\xi)$. Then by the way that S_J is constructed, c.f., Equation (38), there are a positive mass of patients who are willing to accept type-J kidneys and also have their scores strictly larger than s. Since the population distribution at the steady state is stationary, the mass of those patients will stay invariant. Thus, all new arrived type-J kidneys will be consumed by those patients before being offered to the patient with a lower score s. Consequently, the patient would never receive any kidneys of type J before his waiting time has accumulated to $\tau_J(\xi)$.

When the patient has waited for exactly $\tau_J(\xi)$ time units, his score has to be equal or greater than S_J by the "furthermore" part of the proposition that we have proved at the beginning of this proof. Then by the HL property of the score, the patient's score must continue increasing during the time period $[\tau_J(\xi), \tau_J(\xi)+\delta]$ for sufficiently small $\delta > 0$. Thus, immediately after $\tau_J(\xi)$, the patient's score will be strictly larger than S_j, and thus be larger than the scores of all other patients on the waitlist who are willing to accept a type-J kidney. Consequently, the patient will receive a type-J kidney offer either at or immediately after $\tau_J(\xi)$. It is complicated to differentiate these two scenarios. However, for all kidney types $j = 1, \ldots, J$, these two scenarios only result in an infinitesimal difference in the values of $\tau_j(\xi)$ and U_R. Thus, the two scenarios make no difference for a patient's acceptance/rejection decision except when the patient has $U_A = U_R$. However, as we argued earlier, a tie can only happen to patients with a particular η, which has a measure of zero.

Therefore, there is no need to differentiate these two scenarios, and we will simply assume in the remainder of our paper that a patient receives a kidney offer at the allocation time.

After we obtain S_J and $\tau_J(\xi)$, then whenever a patient is offered a type $J-1$ kidney, we can compute his expected QALYs by rejecting that kidney, U_R. For example, if a patient with parameters (ξ,η) is offered a type $J-1$ kidney at time t, then by knowing his waiting time for type-J kidney, $\tau_J(\xi)$, his U_R is exactly given by $V(\tau_J(\xi)-t)$ as defined in Equation (39) with $c=c(t;\xi)$ and $j=J$. Consequently, we can compute almost every patient's optimal strategy $a(\tau,J-1|\xi,\eta)$. We then iteratively repeat the above procedure for $j=J-1,\ldots,1$. In each iteration, by knowing almost every patient's strict dominating strategy $a(\tau,j|\xi,\eta)$, we can compute the score threshold S_j and the required waiting time $\tau_j(\xi)$ for that type of kidneys. This allows us to calculate every patient's U_R and to determine $a(\tau,j-1|\xi,\eta)$, the strictly dominating strategy for the type-$(j-1)$ kidneys for almost all patients (except for those with $U_A=U_R$) given that almost everyone will play the strategy $a(\tau,j|\xi,\eta)$ for the type-j kidneys. Following the above procedure, which is well known as iterated elimination of dominated strategies, we can find the unique Nash equilibrium strategy. While we have proved that the $\tau_j(\xi)$ constructed in the above procedure satisfies the properties stated in the proposition.

Finally, we provide an example to illustrate that there might be not such a score threshold or allocation time when the ranking policy does not satisfy the HL property. Consider a waitlist with a single kidney type, in which the total kidney supply rate μ is lower than the aggregate patient arrival rate λ. Suppose all patients are ranked according to a score $L(c,j,\tau)=-\tau$. This is the donor-blind, last-come-first-serve policy, and does not satisfy the HL property. It is easy to check that patients with scores lower than zero will never be offered a kidney. Among patients whose scores are equal to zero, which only happens at the time of their arrival, only a proportion μ/λ are offered a kidney; while the other patients would never receive a kidney offer. So there is not such a score threshold or allocation time as described in Proposition 1. ■

Appendix B: Proof of Lemma 1

Consider a patient with health score ξ at time 0 (here 0 refers to the current time, not necessarily the time of arrival), and the quality-of-life coefficient η for that patient living on dialysis. We define the function

$$V(t):=\eta\int_0^t sf_\xi(s)\,ds+\eta\bar{F}_\xi(t)t+\bar{F}_\xi(t)c(t;\xi)\phi_j. \tag{39}$$

In the above equation, $\eta\int_0^t sf_\xi(s)\,ds+\eta\bar{F}_\xi(t)t$ and $\bar{F}_\xi(t)c(t;\xi)\phi_j$ gives the expected QALYs before and after the transplantation, respectively. Thus, $V(t)$ gives the optimal expected QALYs that the patient may receive conditional on accepting a type-j kidney offer at time t. We note that

$$V'(t)=\eta\bar{F}_\xi(t)-f_\xi(t)c(t;\xi)\phi_j+\bar{F}_\xi(t)c'(t;\xi)\phi_j\le\bar{F}_\xi(t)-f_\xi(t)c(t;\xi)\phi_j<0, \tag{40}$$

where the first inequality follows from $c'(t;\xi)\le 0$, and the second inequality follows from $\eta<1$ and that

$$1-\frac{f_\xi(t)}{\bar{F}_\xi(t)}c(t;\xi)\phi_j=1-h(c(t;\xi))c(t;\xi)\phi_j<0$$

by (3). Thus, $V(t)$ strictly decreases with t. Consequently, whenever a type j kidney is offered, the patient's strictly dominating strategy is to accept it immediately rather than accepting a kidney of equal or even lower quality at a later time. If a patient chooses to stay on dialysis forever, then it is equivalent to accepting a kidney with $\phi=0$ at an infinitely later time. So this case is also covered by this proof. ■

Appendix C: Proof of Proposition 2 and Remark 1

Proof. We proceed by contradiction. Suppose a patient with parameters with parameters (ξ,η), by using his optimal strategy, rejects a kidney of type j_k at time $\tau_{j_k}(\xi)$, but accepts a kidney of the same type j_k at a later time $s>\tau_{j_k}(\xi)$. After the patient rejects the kidney at time $\tau_{j_k}(\xi)$, let us assume that the first kidney type he will accept is $j_{k'}$. If $j_{k'}\neq j_k$, then the patient would either die before transplantation, or accept a kidney of type j' and leave the waitlist. In whichever case, the patient will not accept a kidney of type j_k, which contradicts our assumption. If $j_k=j_{k'}$, then it is clear from Lemma 1 that accepting the type-j_k kidney earlier at time $\tau_{j_k}(\xi)$ strictly dominates the postulated alternative, and hence, leads to a contradiction. Therefore, we conclude that the patient's choices throughout their waits are consistent and it suffices to restrict attention to decision times $\tau_{j_1}(\xi),\dots,\tau_{j_m}(\xi)$.

We next prove that $j^*(\eta,\xi)$ is non-decreasing in η by contradiction. Suppose there are two patients with $\eta_1>\eta_2$, but $j_1(:=j^*(\xi,\eta_1))<j_2(:=j^*(\xi,\eta_2))$. Then kidney type j_1 must have been accepted by the first patient, and rejected by the second one; note that j_2 is the first kidney type accepted by the second patient subsequent to the offer of type j_1. Given that the second patient rejects kidney type j_1 and accepts j_2, we can derive the following inequality by recursively applying formula (10),

$$c(\tau_{j_1}(\xi);\xi)\phi_{j_1}<\eta_2\int_{\tau_{j_1}(\xi)}^{\tau_{j_2}(\xi)}\frac{f_\xi(t)}{\bar F_\xi(\tau_{j_1}(\xi))}(t-\tau_{j_1}(\xi))\,dt+\frac{\bar F_\xi(\tau_{j_2}(\xi))}{\bar F_\xi(\tau_{j_1}(\xi))}(\eta_2(\tau_{j_2}(\xi)-\tau_{j_1}(\xi))+c(\tau_{j_2}(\xi);\xi)\phi_{j_2}). \quad (41)$$

Intuitively, the above inequality implies that for the second patient, choosing to wait till the offer of kidney type j_2 strictly dominates accepting kidney type j_1. Since $\eta_1>\eta_2$, the above inequality still holds by replacing η_2 with η_1. However, that implies that the first patient has to reject kidney j_1, which contradicts that $j^*(\xi,\eta_1)=j_1$. This concludes the proof. ■

Remark 1 *Since $j^*(\eta,\xi)$ can be computed from $(\tau_j(\xi))$ using the recursive equations* (10)-(12) *and* (13)*, we can then describe the equilibrium state using only $(\tau_j(\xi))$ instead of the waitlist density* Π*. In fact, given $(\tau_j(\xi))$, we can compute each patient's matched kidney type $j^*(\eta,\xi)$. That allows us to determine if a certain patient has already accepted a kidney offer and left by comparing the patient's current waiting time τ and his allocation time $\tau_{j^*(\eta,\xi)}(\xi)$. Specifically, for a patient with health score c and waiting time τ, we can solve his initial health score ξ from the equality $c=c(\tau;\xi)$. We can then recover $\Pi:=(\pi_{c,\tau,\eta})$ as follows*

$$\pi_{c,\tau,\eta}=\lambda\,\rho(\xi)\,\bar F_\xi(\tau)\,\chi(\tau<\tau_{j^*(\eta,\xi)}(\xi)). \quad (42)$$

Appendix D: Proof of Proposition 3

Proof. We will only prove the "if" part. The "only if" part can be proved with a similar argument.

Suppose $(\tau_j(\xi))$ are the allocation times at the equilibrium. We discuss two possible situations that could happen to each of the virtual queues. Recall that the j^{th} virtual queue consists of patients who will be matched to kidneys of type j.

In the first situation, queue j is non-empty and thus $z_j=\sup_\xi \tau_j(\xi)>0$. Since it is at the steady state, the inflow and out-flow rates must be balanced for each virtue queue, otherwise the queue-length cannot stay invariant. Note that the departure rate includes those having reneged as well as those having transplanted kidneys of type j. This leads to the following equality,

$$\lambda \int_{\underline{\xi}}^{\bar{\xi}} Q_j(\xi)\rho(\xi)\, d\xi = \lambda \int_{\underline{\xi}}^{\bar{\xi}} F_\xi(\tau_j(\xi))\rho(\xi)\, d\xi + \mu_j, \tag{43}$$

where $\lambda \int_{\underline{\xi}}^{\bar{\xi}} F_\xi(\tau_j(\xi))\rho(\xi)\, d\xi$ gives the aggregate reneging rate in queue j. If we define y_j as in (17), then the above equality implies that $y_j = 0$, and thus the complementary slackness condition $y_j z_j = 0$ holds for j.

In the second situation, queue j is empty, then $\tau_j(\xi) = 0$ for all ξs. Thus, we have $z_j = \sup_\xi \tau_j(\xi) = 0$ for patients in queue j, and the complementary slackness condition holds. If a queue is empty, there can be a surplus in the kidney and thus y_j can be any non-negative number. So the constraint $y_j \geq 0$ in the NCP holds for that j.

The discussion of the above two situations show that the NCP (17) is solved by $(\tau_j(\xi))$ and the associated variables.

Appendix E: Proof of Lemma 2

Proof. (a): Since $\tau_j(\xi)$ is the first time that the patient's score is equal to or larger than S_j, $\tau_j(\xi)$ must be non-decreasing in S_j. Since $L(c_\xi(\tau), j) + \tau$ is right-continuous in τ, $\tau_j(\xi)$ must be right-continuous. Its left-limit exists due to monotonicity.

(b): By the assumption of the matching policy, a patient with initial score ξ has his score $L(c(t;\xi), j, t)$ strictly increasing in t almost everywhere. At all those points, S_j strictly increases in t and thus $\tau_j(\xi)$ continuously increases in S_j. However, when the patient's health score hits one of the cutoff values in $\mathcal{K}$, say, $\tilde{c}$, the function value $L(\cdot, j, t)$ can possibly take a downward jump at $\tilde{c}$, at which time $\tau_j(\xi)$ changes discontinuously with S_j. To see that, we refer the readers to Figure 2 –when the threshold S_j^1 approaches to $S_j^2 := \lim_{c\uparrow\tilde{c}} L(c, j, c^{-1}(\tilde{c};\xi))$ from left, the allocation time corresponding to S_j^1 for patient ξ, i.e., $\tau_j^1(\xi)$, does not to approach to $c^{-1}(\tilde{c};\xi)$ from left. This is because the $L(c(t;\xi), j, t)$ has a downward jump at $\tilde{c}$. Consequently, it takes much longer for the score of patient ξ to reach S_j^2. That implies that the allocation time $\tau_j(\xi)$ changes discontinuously at S_j^2. However, for any given S_j^2, since both $c^{-1}(c;\xi)$ is strictly increasing in ξ and $L(c, j, t)$ is strictly increasing in t, the left-hand-side of the following equation is strictly increasing in ξ and the equation must have at most one solution ξ for any given S_j^2 and $\tilde{c} \in \mathcal{K}$,

$$\lim_{c\uparrow\tilde{c}} L(c, j, c^{-1}(c;\xi)) = S_j^2. \tag{44}$$

Because the above equation is a necessary condition for $\tau_j(\xi)$ to be discontinuous at S_j^2, there is at most one ξ such that $\tau_j(\xi)$ is discontinuous at S_j^2. Since the set $\mathcal{K}$ contains finitely many points, there are finitely many ξs at which $\tau_j(\xi)$ may change discontinuously in S_j, and those ξs have a measure of zero. ■

Appendix F: Proof of Theorem 1

Proof. Let $S := (S_j)$ denote a vector of score thresholds for all kidney types. We next construct a mapping $\Psi(\cdot)$ to serve the following purpose: if S is a fixed point of this mapping (that means, $S = \Psi(S)$), then its associated allocation times $(\tau_j(\xi))$ must solve the NCP, c.f., Equation (17), and thus be the equilibrium allocation times. Thus, to prove the theorem, it suffices to prove that such a fixed point exists and is unique.

The mapping $\Psi(\cdot)$ is constructed according to the following steps. Given (S_j), we calculate the unique allocation times $\tau_j(\xi)$ for each patient ξ and j using Equation (8), and then find functions $\Gamma_j(\cdot)$ and $Q_j(\cdot)$ that are associated with $(\tau_j(\xi))$. Finally, with $Q_j(\cdot)$, we search for another vector of score thresholds $\tilde{S} := (\tilde{S}_j)$ and the allocation times $(\tilde{\tau}.(\cdot))$ associated with $\tilde{S}$, such that $\tilde{\tau}.(\cdot)$ and its associated $Q_j(\cdot)$, $\Gamma_j(\cdot)$ solve the NCP (17). If this requirement can be satisfied by multiple $\tilde{S}$s, then we let $\Psi(S)$ be the infimum of them.

We next show that the mapping $\Psi(\cdot)$ constructed above satisfy the desired properties, which allows us to invoke the Brouwer's fixed point theorem and prove the existence of the fixed point. First, the image of $\Psi(\cdot)$ must be contained in the compact set $\prod_{j=1}^{J}[0, \bar{S}_j]$, where $\bar{S}_j$ denote the upper limit of a patient's score for kidney type j,

$$\bar{S}_j := \sup\{L(c(t;\xi), j, t) | \underline{\xi} \leq \xi \leq \bar{\xi},\, 0 \leq t \leq \bar{\tau}_\xi\}. \tag{45}$$

Thus, the mapping from S to $\tilde{S}$ is from the compact domain $\prod_{j=1}^{J}[0, \bar{S}_j]$ to itself (not necessarily onto).

Second, $\Psi(\cdot)$ is well-defined. That means, we can always find a $\tilde{S}$ such that its associated allocation times $(\tilde{\tau}_j(\xi))$ solve the NCP, or equivalently, for all $j = 1, ldots, J$, $(\tilde{\tau}_j(\xi))$ either solves the following identity,

$$\lambda \int_{\xi \in \Omega} Q(\xi, j)\rho(\xi)\bar{F}_\xi(\tilde{\tau}_j(\xi))\, d\xi = \mu_j, \tag{46}$$

or $\tilde{\tau}_j(\xi) = 0$ for all ξ and the following inequality holds,

$$\lambda \int_{\xi \in \Omega} Q(\xi, j)\rho(\xi)\, d\xi \leq \mu_j. \tag{47}$$

The detailed proof follows next. For each j, by Property (b) of Lemma 2, $\tilde{\tau}_j(\xi)$ increases continuously in $\tilde{S}_j$ for almost all ξs. We can deduce that the integral on the left-hand-side of Equation (46) has to decrease continuously with $\tilde{S}_j$. Note that if the threshold $\tilde{S}_j$ is set as low as $\min_\xi L(\xi, j, 0)$, it does not require any patient to wait for kidneys of type j so the associated allocation time $\tilde{\tau}_j(\xi) = 0$ for all ξ; if the threshold $\tilde{S}_j$ is set as its upper limit $\bar{S}_j$, then all patients must have reneged before their allocation times by the definition of $\bar{S}_j$. Therefore, when $\tilde{S}_j$ increases from $\min_\xi L(\xi, j, 0)$ to $\bar{S}_j$, the left-hand-side of Equation (46) continuously decreases from $\lambda \int_{\xi \in \Omega} Q_j(\xi)\rho(\xi)\, d\xi$ (which is the function value corresponding to $\tau_j(\xi) = 0$ for all ξ) to 0. Thus, either $\mu_j \in [0, \lambda \int_\xi Q(\xi, j)\rho(\xi)\, d\xi]$, in which case the intermediate value theorem implies the existence of a solution to equality (46), or μ_j is outside this interval, in which case inequality (47) must hold. Note that in the first case, it is possible that Equation (46) has multiple solutions. This could happen when the curve $L(c(\tau;\xi), j, \tau)$ takes an upward jump at $\tilde{\tau}_j(\xi)$. Then all $\tilde{S}_j \in [\lim_{\tau \uparrow \tilde{\tau}_j(\xi)} L(c(\tau;\xi), j, \tau), L(c(\tilde{\tau}_j(\xi);\xi), j, \tilde{\tau}_j(\xi))]$ are associated with the same allocation time $\tilde{\tau}_j(\xi)$. We then let $(\Psi(S))_j = \lim_{\tau \uparrow \tilde{\tau}_j(\xi)} L(c(\tau;\xi), j, \tau)$, which is associated with $\tilde{\tau}_j(\xi)$ by right continuity of the function $L(c(\tau;\xi), j, \tau)$ (i.e., Property (a) of Lemma 2).

Finally, we show that $\Psi(\cdot)$ is continuous, that is, $\|\Psi(S) - \Psi(S')\| \to 0$ if $\|S - S'\| \to 0$, where $\|\cdot\|$ denote the supremum norm. We will use the superscript $'$ to denote the variables associated with S'. For example, $\tau_j(\xi)$ and $\tau'_j(\xi)$ denote the allocation times associated with S and S', respectively. By Property (b) of Lemma 2, we have $\|\tau_j(\xi) - \tau_{j'}(\xi)\| \to 0$ for almost all ξ. Then the recursive equations (10) imply that $|V_k - V'_k| \to 0$ for for almost all ξ and all k, where V_k denotes the optimal expected QALYs that a patient can get from the time of receiving kidney offers of type j_k. Since V_k and V'_k stay very close for almost all ξ, by changing V to V', only the patients with η near the cutoff values in $\{\Gamma_j(\xi) \,|\, j = 1, \ldots, J-1\}$ may change their matched kidney type $j^*(\xi,\eta)$. Therefore, when $|V_k - V_k'| \to 0$, for almost all ξ, $|Q_j(\xi) - Q_j'(\xi)| \to 0$. Consequently, the left-hand-side of (46), i.e., $\lambda \int_{\xi\in\Omega} Q_j(\xi)\rho(\xi)\bar{F}_\xi(\tau_j(\xi))\,d\xi$ continuously decreases with S. According to the previous discussion, $\tilde{S}$ is either the constant $\min_\xi L(\xi, j)$, at which $\tau_j(\xi) = 0$ for all ξs, or the unique intermediate solution to (46). In the first case, $\tilde{S}$ is a constant and thus changes continuously with S; in the second case, since the derivative of the left-hand-side of (46) with respect to $\tilde{S}$ is strictly negative, a small perturbation to the left-hand-side led by replacing S with S', can result in only small perturbation of its solution $\tilde{S}$. That implies $|\tilde{S} - \tilde{S}'| \to 0$.

We thus proved the existence of a fixed point $S = \Psi(S)$. We next prove its uniqueness by contradiction. Suppose there are two different allocation-time vectors, $\tau_j(\xi)$ and $\tau_j'(\xi)$, which are associated with two different score-thresholds vectors, S and S'. Define the index set

$$\mathcal{J}^+ := \{j \,|\, S_j < S_j'\}. \tag{48}$$

Intuitively, $\mathcal{J}^+$ contains kidney types for which the allocation time at the first equilibrium is not longer than that at the second equilibrium. Since $S \neq S'$, we can assume $\mathcal{J}^+ \neq \emptyset$ without loss of generality. We note that if a patient has $j^*(\xi,\eta) \in \mathcal{J}^+$ under S, then she must still have $j^*(\xi,\eta) \in \mathcal{J}^+$ if the score thresholds S' was replaced by S, because the latter requires an even shorter waiting time for those kidney types. Thus, the total mass of patients will choose kidney types in $\mathcal{J}^+$ under S is no less than that under S'. By property (a) of Lemma 2, we have $\tau_j(\xi) \le \tau_j'(\xi)$ for all ξ and all $j \in \mathcal{J}^+$. Since $(\tau_j(\xi))$ and $(\tau_j'(\xi))$ are different allocation-time vectors, we must have $\tau_j(\xi) < \tau_j'(\xi)$ for some j. Without loss of generality, we assume $j \in \mathcal{J}^+$ (otherwise we can define $\mathcal{J}^+$ by swapping S and S' to have $j \in \mathcal{J}^+$). That implies the total mass of patients who die during their waiting for kidney types in $\mathcal{J}^+$ is larger under S' compared to that under S, despite the fact that the patients who choose to accept kidney types in $\mathcal{J}^+$ under S' is a subset of that under S. Consequently, the total mass of patients who will transplant kidneys of types in $\mathcal{J}^+$ under S is strictly larger than that under S'. However, since $S_j' > S_j \ge 0$ for all $j \in \mathcal{J}^+$, no queue in $\mathcal{J}^+$ is empty under S', which implies that the utilization of kidneys in classes $\mathcal{J}^+$ have already been 100% under S', which contradicts that more kidneys in classes $\mathcal{J}^+$ are accepted under S. ■

Appendix G: Proof of Lemma 3

Because $\boldsymbol{\tau}$ satisfies constraint (C.M), $\gamma_j(\xi)$ must be non-decreasing in ξ. Consequently, $\gamma_j(\xi)$ must have both left and right limits. We next prove that $\gamma_j(\xi)$ is left-continuous. Suppose a sequence of values $\{\xi^k\} \uparrow \xi^*$. Since $\gamma_j(\xi^k)$ is non-decreasing, we have $\{\gamma_j(\xi^k)\} \uparrow c^*$ for some c^*, which further implies that $\{\tau_j(\xi^k)\} \to c^{-1}(c^*;\xi^*)$. Because $L(c, j, \tau)$ is continuous in τ and left-continuous in c for all j, we have

$$L(\gamma_j(\xi^k), j, \tau_j(\xi^k)) \to L(c^*, j, c^{-1}(c^*;\xi^*)) \text{ when } k \to \infty. \tag{49}$$

Since for each k, $\tau_j(\xi^k)$ is the allocation time of patient ξ^k, Proposition 1 implies that $L(\gamma_j(\xi^k), j, \tau_j(\xi^k)) \ge S_j$. Consequently, $L(c^*, j, c^{-1}(c^*;\xi^*)) \ge S_j$. That implies $c^{-1}(c^*;\xi^*) \le \tau_j(\xi^*)$, so

$$c^* \ge c(\tau_j(\xi^*);\xi^*) = \gamma_j(\xi^*), \tag{50}$$

On the other hand, monotonicity of $\gamma_j(\cdot)$ implies that $c^* = \lim_{k\to\infty} \gamma_j(\xi^k) \le \gamma_j(\xi^*)$. We thus deduce that $c^* = \lim_{k\to\infty} \gamma_j(\xi^k) = \gamma_j(\xi^*)$. Thus, $\gamma_j(\cdot)$ is left-continuous.

Finally, we prove that $\gamma^{-1}(c)$ is a singleton for $c \notin \mathcal{K}$. Suppose $\xi, \xi' \in \gamma^{-1}(c)$. Then since $c \notin \mathcal{K}$, we have $L(c, j, \tau_j(\xi)) = L(c, j, \tau_j(\xi')) = S_j$ by Proposition 1. Since $L(c, j, \tau)$ strictly increases in τ for all c and j, we deduce that $\tau_j(\xi) = \tau_j(\xi')$. That implies that $\xi = \xi'$ as $c(\tau_j(\xi); \xi) = c(\tau_j(\xi'); \xi')$. ■

Appendix H: Proof of Proposition 2

Proof. To prove the "$\subseteq$" direction in Equation (21), it suffices to prove that if $\boldsymbol{\tau}$ has a minimum acceptance level k, then $\boldsymbol{\tau} \in \mathcal{P}_k$. We have argued in Section 3 that $\boldsymbol{\tau}$ must satisfy constraint (C.1)-(C.6) by the properties of the minimum acceptance level. We next prove that $\boldsymbol{\tau}$ satisfies constraint (C.M) by contradiction. Suppose two patients have their initial scores $\xi > \xi'$, but $c(\tau_j(\xi); \xi) < c(\tau_j(\xi'); \xi')$. Since $c(\cdot; \xi)$ is continuously decreasing, there exists a time $t < \tau_j(\xi)$, such that $c(t; \xi) = c(\tau_j(\xi'); \xi')$. Then by $\xi > \xi'$, we deduce that $t > \tau_j(\xi')$. Since $\partial L / \partial \tau > 0$, we have

$$L(c(t;\xi), j, t) = L(c(\tau_j(\xi'); \xi'), j, t) > L(c(\tau_j(\xi'); \xi'), j, \tau_j(\xi')) \geq S_j \tag{51}$$

where the last inequality follows from Proposition 1. The above equality implies that at time $t < \tau_j(\xi)$, the score of patient ξ for kidney type j is already strictly greater than S_j, which contradicts with that the patient's score firstly reaches or exceeds S_j at time $\tau_j(\xi)$.

To prove the "$\supseteq$" direction, it suffices to prove that for $k = 0, 1, \ldots, J$, any $\boldsymbol{\tau} \in \mathcal{P}_k$ is a solution to the NCP, c.f., (17), and thus is an equilibrium allocation-time vector by Proposition 3. Let $y_j = \mu_j - \lambda \int_{\underline{\xi}}^{\bar{\xi}} Q_j(\xi) \bar{F}_\xi(\tau_j(\xi)) \rho(\xi) \, d\xi$ and $z_j = \sup\{\tau_j(\xi) \,|\, \xi \in [\underline{\xi}, \overline{\xi}]\}$ following their definitions in the NCP. The inequality constraints $z_j \geq 0$ follows from the non-negative constraints for $\tau_j(\xi)$ in (C.6). The inequality constraint $y_j \geq 0$ follows from (C.2) for $j > k$, and (C.3) for $j = k$. For $j < k$, (C.4) implies $Q_j = 0$. Thus, $y_j = \mu_j \geq 0$. It remains to prove the complementary slackness condition $y_j z_j = 0$ for all j. For $j \leq k$, constraint (C.5) implies $z_j = 0$, which leads to the complementary slackness condition. For $j > k$, constraint (C.6) implies $y_j = 0$ and thus the complementary slackness condition. Thus, $(\tau_j(\xi))$ is an equilibrium allocation time.

We next prove that any $\boldsymbol{\tau} \in \mathcal{P}_k$ can be achieved by a score $L(c, j, \tau) = M(c, j) + \tau$ for score threshold $S_j = 0$ $(j = 1, \ldots, J)$, where $M(c, j)$ was constructed as in Equation (23) the the theorem. To that end, it suffices to show that $(\tau_j(\xi))$ solves Equation (8) for all j and ξ,. Because $S_j = 0$, Equation (8) reduces to the following equality

$$\tau_j(\xi) := \begin{cases} \bar{\tau}_\xi, & \text{if } \{\tau \geq 0 : M(c(\tau;\xi), j) + \tau \geq 0\} = \emptyset, \\ \min\{\tau \geq 0 : M(c(\tau;\xi), j) + \tau \geq 0\}, & \text{otherwise.} \end{cases} \tag{52}$$

Note that when the set is empty, i.e., a patient could never reach the threshold 0, we let his allocation time to be $\bar{\tau}_\xi$ instead of $+\infty$ to make the allocation time bounded. That does not make a difference because a patient with initial health score ξ cannot live no longer than $\bar{\tau}_\xi$.

We next prove Equation (52). It suffices to prove that $L(c(\tau_j(\xi); \xi), j, \tau_j(\xi)) \geq 0$, and $L(c(t; \xi), j, t) < 0$ for all $t < \tau_j(\xi)$. To prove the first inequality, we note that

$$L(\gamma_j(\xi), j, \tau_j(\xi)) = M(\gamma_j(\xi), j) + \tau_j(\xi) = -\inf\{\tau_j(z) \,|\, z \in \gamma_j^{-1}(\gamma_j(\xi))\} + \tau_j(\xi) \geq 0, \tag{53}$$

where the second equality follows the definition of $M(c, j)$ in Equation (23).

To prove the second inequality, we discuss the following two cases. If $c(t;\xi) \notin \mathcal{C}$, then by the definition of $M(c,j)$, we have

$$L(c(t;\xi),j,t) = M(c(t;\xi),j) + t = -\bar{\tau}_{\bar{\xi}} + t < 0, \tag{54}$$

as no patient can live longer than $\bar{\tau}_{\bar{\xi}}$. If $c(t;\xi) \in \mathcal{C}_j$, then by definition of $\mathcal{C}_j$, the set $\gamma_j^{-1}(c(t;\xi)) := \{z|\gamma_j(z) = c(t;\xi)\}$ is non-empty. For any $z \in \gamma_j^{-1}(\xi)$, since $t < \tau_j(\xi)$, we have $\gamma_j(\xi) = c(\tau_j(\xi);\xi) < c(t;\xi) = \gamma_j(z)$, that implies $\xi < z$ by monotonicity of $\gamma_j(\cdot)$. Because $\xi < z$ but $c(t;\xi) = c(\tau_j(z);z)$, we deduce that $t < \tau_j(z)$. Therefore,

$$L(c(t;\xi),j,t) = M(c(t;\xi),j) + t < M(c(\tau_j(z);z),j) + \tau_j(z) = L(c(\tau_j(z);z),j,\tau_j(z)). \tag{55}$$

If $c(t;\xi) \notin \mathcal{K}$, then Proposition 1 implies that $L(c(\tau_j(z);z),j,\tau_j(z)) = S_j = 0$; so the above inequality implies $L(c(t;\xi),j,t) < 0$; if $c(t;\xi) = \gamma_j(z) \in \mathcal{K}$, then since the above inequality holds for all $z \in \gamma_j^{-1}(c(t;\xi))$, we have

$$L(c(t;\xi),j,t) \le \inf\{L(\gamma_j(z),j,\tau_j(z))\,|\,z \in \gamma_j^{-1}(c(t;\xi))\} = M(\gamma_j(z),j) + \inf\{\tau_j(z)\,|\,z \in \gamma_j^{-1}(c(t;\xi))\} = 0, \tag{56}$$

where the last equality follows from $\inf\{\tau_j(z)\,|\,z \in \gamma_j^{-1}(c(t;\xi))\} = -M(c(t;\xi),j) = -M(\gamma_j(z),j)$. Therefore, regardless of $c(t;\xi) \in \mathcal{K}$ or $c(t;\xi) \notin \mathcal{K}$, we have proved that $L(c(t;\xi),j,t) < 0$ for all $t < \tau_j(\xi)$. We have thus proved that $\tau_j(\xi)$ is the first time that a patient's score reaches 0 and verified Equation (52).

It remains to prove that the score function $L(c,j,\tau) = M(c,j) + \tau$ satisfies the assumptions of a matching policy. It suffices to show that $M(c,j)$ is p.c.d. for all j, and

$$\frac{dM(c(\tau;\xi),j)}{d\tau} > -1 \text{ for all } j,\xi \text{ when } c(\tau;\xi) \notin \mathcal{K}. \tag{57}$$

To prove that $M(c,j)$ is p.c.d., for all $c \notin \mathcal{K}$, we note that $\gamma_j^{-1}(c)$ is a singleton by Lemma 3. Furthermore, since $\gamma_j(\xi)$ is left-continuous and non-decreasing in ξ, $\mathcal{C}_j$ must have the following form,

$$\mathcal{C}_j = [\underline{c}_{j1}, \bar{c}_{j1}] \cup (\underline{c}_{j2}, \bar{c}_{j2}] \cup \ldots \cup (\underline{c}_{jm}, \bar{c}_{jm}], \tag{58}$$

where m is a positive integer and the cutoff points satisfy $\bar{c}_{i-1,j} < \underline{c}_{ij} \le \bar{c}_{ij}$ for $i = 2,3,\ldots,m$. Therefore, over each interval $(\underline{c}_{j\ell}, \bar{c}_{j\ell}]$ with $\bar{c}_{j\ell} > \underline{c}_{j\ell}$ (let $(\underline{c}_{ij}, \underline{c}_{ij}] = \{\underline{c}_{ij}\}$ by abuse of notation). $d\gamma_j^{-1}(c)/dc = 1/(\gamma_j'(\gamma_j^{-1}(c)))$ exists due to the assumption that $\tau_j(\xi)$ (and therefore $\gamma_j(\xi)$) is p.c.d. in ξ. Since $\mathcal{K}$ contains finite points, $M(c,j) = \tau_j(\gamma^{-1}(c))$ is p.c.d.

We next prove condition (57). Suppose $c = c(\tau;\xi) \notin \mathcal{K}$. For each $j = 1,\ldots,J$, if $c \notin \mathcal{C}_j$, then $\partial M(c(\tau;\xi),j)/\partial c = 0$ implies $dM(c(\tau;\xi),j)/d\tau = 0$, and condition (57) is proved; if $c \in \mathcal{C}_j$, then by Lemma 3, $\gamma_j^{-1}(c)$ is a singleton. Thus, the following equation holds for all $c \in \mathcal{C}_j$,

$$c = c(\tau_j(\gamma_j^{-1}(c))\,;\,\gamma_j^{-1}(c)) = H^{-1}\left(\tau_j(\gamma_j^{-1}(c)) + H(\gamma_j^{-1}(c))\right). \tag{59}$$

where the first equality follows from the definition of the inverse function $\gamma_j^{-1}(\cdot)$, and the second equality follows from Equation (2). We have argued earlier that the derivative $\gamma_j^{-1}(c)$ exists over each sub-interval of $\mathcal{C}_j$. Thus, by taking derivative at both sides of Equation (59), we have

$$\begin{aligned} 1 &= \frac{dH^{-1}(t)}{dt}\Big|_{t=\tau_j(\gamma_j^{-1}(c))+c^{-1}(\gamma_j^{-1}(c))} \left(\frac{d\tau_j(\gamma_j^{-1}(c))}{dc} + \frac{dH(\gamma_j^{-1}(c))}{dc}\right) \\ &= \frac{1}{H'(c)}\left(\frac{d\tau_j(\gamma_j^{-1}(c))}{dc} + \frac{dH(\gamma_j^{-1}(c))}{dc}\right) \\ &> \frac{1}{H'(c)}\frac{d\tau_j(\gamma_j^{-1}(c))}{dc}. \end{aligned} \tag{60}$$

The second equality follows from that for all t and ξ, the derivative $\frac{dH^{-1}(t)}{dt}$ only depends on the patient's up-to-date health score $c = H^{-1}\left(\tau_j(\gamma_j^{-1}(c)) + H(\gamma_j^{-1}(c))\right)$. For the last inequality, we note that $\gamma_j(\cdot)$ is strictly increasing, so the inverse function $\gamma_j^{-1}(c)$ is strictly increasing in c. Since $H(\cdot)$ is strictly decreasing, we deduce that $dH(\gamma_j^{-1}(c))/dc < 0$, which, together with $H'(c) < 0$, lead to the last inequality in (60). As a result, for all $c \in \mathcal{C}_j \backslash \mathcal{K}$, we have

$$\frac{dM(c,j)}{d\tau} = \frac{\partial M(c,j)}{\partial c} c'(\tau\,;\gamma^{-1}(c)) = -\frac{d\tau_j(\gamma_j^{-1}(c))}{dc}\frac{1}{H'(c)} > -1, \tag{61}$$

where the second equality follows from the definition of $M(c,j)$ on $c \in \mathcal{C}_j$, and the inequality follows from (60). Thus, for each $j = 1, \ldots, J$, we have proved condition (57) when $c \in \mathcal{C}_j$. This completes the proof. ■

Appendix I: An Example Showing that $\mathcal{P}_{DB}$ Cannot be Recovered by Affine Scores

Here we present an example to illustrate that the achievable region of donor-blind policies cannot be recovered by scores in the form of $\tilde{M}(C) + \tau$. Suppose two patients with initial health scores ξ^1 and ξ^2 satisfy $c(\tau_1(\xi^1);\xi^1) = c(\tau_2(\xi^2);\xi^2) \notin \mathcal{K}$, that is, the first patient, when being offered a kidney of type 1, has exactly the same health score as that of the second patient when being offered a kidney of type 2. We further assume that there are two other patients with initial health scores ξ^3 and ξ^4 such that $c(\tau_1(\xi^3);\xi^3) = c(\tau_2(\xi^4);\xi^4) \notin \mathcal{K}$. Then if the donor-blind score has a form $\tilde{M}(c) + \tau$, then by Proposition 1 we have

$$\begin{aligned} S_1 &= \tilde{M}(c(\tau_1(\xi^1);\xi^1)) + \tau_1(\xi^1) = \tilde{M}(c(\tau_1(\xi^3);\xi^3)) + \tau_1(\xi^3) \\ S_2 &= \tilde{M}(c(\tau_2(\xi^2);\xi^2)) + \tau_1(\xi^2) = \tilde{M}(c(\tau_2(\xi^4);\xi^4)) + \tau_2(\xi^4), \end{aligned} \tag{62}$$

As $c(\tau_1(\xi^1);\xi^1) = c(\tau_2(\xi^2);\xi^2)$, $c(\tau_1(\xi^3);\xi^3) = c(\tau_2(\xi^4);\xi^4)$, we have the following equality

$$\tau_1(\xi^1) - \tau_1(\xi^3) = \tau_1(\xi^2) - \tau_1(\xi^4). \tag{63}$$

The above equality is not implied by any constraints in the expression of $\mathcal{P}_{DB}$. That means, if we assume the score of a donor-blind policy to take the special form of $\tilde{M}(c) + \tau$, then we have implicitly imposed an extra equality constraint (63). Thus, any allocation time that violates equality (63) cannot be achieved by the score $\tilde{M}(c) + \tau$, though it could always be achieved by a score in its general form. In fact, the above example also suggests that other special forms such as $L(c,\tau) = \tilde{M}(c) + \hat{M}(\tau)$ or $L(c,\tau) = \tilde{M}(c)\hat{M}(\tau)$ cannot recover the entire achievable region for similar reasons.

Appendix J: Numerical Procedure for Solving the Policy Design Problem

To facilitate real-time computation, we propose a finite-dimensional approximation of (31) by discretizing the domain of ξ into N grid points: $\{\ell(\overline{\xi} - \underline{\xi})/N \mid \ell = 0, 1, \ldots, N\}$. The finite-dimensional optimization problem then searches for an $(N+1)$-by-J matrix of the allocation times: $\boldsymbol{\tau}^f := \{\tau_j(\xi_\ell) \mid j = 1, \ldots, J,\ \ell = 0, \ldots, N\}$, where $\xi_\ell := \ell(\overline{\xi} - \underline{\xi})/N$ and the superscript f stands for "finite-dimensional". After obtaining $\boldsymbol{\tau}^f$, we recover the continuous allocation time using linear interpolation.

We provide a finite-dimensional representation for the achievable region of the matching policies below.

$$\mathcal{A}_M^f := \cup_{k=0}^J \mathcal{P}_k^f \cap \mathcal{P}_M^f,$$

where

$$\mathcal{P}_k^f := \left\{ \boldsymbol{\tau}^f \in \mathbb{R}_+^{N+1,J} \;\middle|\; \begin{array}{ll} p_j(\xi_\ell) = \frac{1}{\mu_j}\lambda Q_j(\xi_\ell;\boldsymbol{\tau}^f)\rho(\xi_\ell)\bar{F}_\xi(\tau_j(\xi_\ell)) \text{ for all } j,\ell & \text{(Cf.1)} \\ \sum_{\ell=1}^N p_j(\xi_\ell) = 1 \text{ for } j > k & \text{(Cf.2)} \\ \sum_{\ell=1}^N p_k(\xi_\ell) \le 1 & \text{(Cf.3)} \\ Q_j(\xi_\ell;\boldsymbol{\tau}^f) = 0 \text{ for } j < k, \text{ all } \ell & \text{(Cf.4)} \\ \tau_j^f(\xi_\ell) = 0 \text{ for } j \le k, \text{ all } \ell & \text{(Cf.5)} \\ 0 \le \tau_j^f(\xi_\ell) \le \bar{\tau}_{\xi_\ell} \text{ for } j > k, \text{ all } \ell & \text{(Cf.6)} \end{array} \right\}.$$

$$\mathcal{P}_M^f := \left\{ \text{p.c.d. functions } \boldsymbol{\tau} \mid c(\tau_j(\xi_\ell);\xi_\ell) \ge c(\tau_j(\xi_{\ell'});\xi_{\ell'}) \text{ for all } \ell > \ell' \qquad \text{(Cf.M)} \right\}.$$

Similarly, to derive a finite-dimensional representations for $\mathcal{A}_{HF}^f$, we just need to replace (Cf.M) with a stronger constraint as follows,

$$\mathcal{P}_{HF}^f := \left\{ \text{p.c.d. functions } \boldsymbol{\tau}^f \mid \tau_j^f(\xi_\ell) \le \tau_j^f(\xi_{\ell'}) \text{ for all } \ell > \ell' \qquad \text{(Cf.HF)} \right\}.$$

For the donor-blind policies, we need derive a finite-dimensional representation of constraint (C.DB). In particular, this calls for a finite-dimensional representation for the function $L(c(\tau;\xi),\tau)$ which has a continuous domain $\{(\xi,\tau) | \xi \in [\underline{\xi},\bar{\xi}],\ \tau \in [0,\bar{\tau}_\xi]\}$. For that purpose, we construct the grid $\{\xi_\ell | \ell = 1,\dots,N\} \otimes \{\tau_r := r\bar{\tau}_\xi/R | r = 0,1,\dots,R\}$ (let $\tau_0 = 0$) on the continuous domain, and represent the function using its values at the grids $\{L(c(\tau_r;\xi_\ell),\tau_r) \mid \ell = 1,\dots,N,\ r = 0,1,\dots,R\}$. We then recover its values on the continuous domain by linear interpolation. This leads to the finite-dimensional representation of (C.DB) as follows

$$\mathcal{P}_{DB}^f := \left\{ \boldsymbol{\tau}^f \in \mathbb{R}_+^{N+1,J} \;\middle|\; \begin{array}{ll} \tau_j^f(\xi_\ell) := \min\{\bar{\tau}_{\xi_\ell}, \min\{\tau \ge 0 : L(c(\tau;\xi_\ell),\tau) \ge S_j\}\} \text{ for some } L(c,\tau) & \\ \text{such that } L(c(\tau_{r+1}^f;\xi_\ell),\tau_{r+1}^f) - L(c(\tau_r^f;\xi_\ell),\tau_r^f) \ge \epsilon \text{ for all } \ell, r & \text{(Cf.DB)} \\ \qquad L(c(\tau_r^f;\xi_\ell),\tau_{r+1}^f) - L(c(\tau_r^f;\xi_\ell),\tau_r^f) \ge \epsilon \text{ for all } \ell, r & \end{array} \right\}$$

Given $\{L(c(\tau_r;\xi_\ell),\tau_r) \mid \ell = 1,\dots,N,\ r = 0,1,\dots,R\}$, we solve $\tau_j^f(\xi_\ell)$ from the first equality in (Cf.DB) as follows. First, we can find the smallest index r such that $L(c(\tau_r;\xi_\ell),\tau_r) \ge S_j$, if such an r exists; otherwise, assign $\tau_j^f(\xi_\ell) = \bar{\tau}_{\xi_\ell}$. Second, because the values of $L(c(\tau;\xi_\ell),\tau)$ on the continuous domain are assigned using linear interpolation, we can locate $\tau_j^f(\xi_\ell)$ as

$$\tau_j^f(\xi_\ell) = \tau_{r-1}^f + \frac{S_j - L(c(\tau_{r-1}^f;\xi_\ell),\tau_{r-1}^f)}{L(c(\tau_r^f;\xi_\ell),\tau_r^f) - L(c(\tau_{r-1}^f;\xi_\ell),\tau_{r-1}^f)}(\tau_r^f - \tau_{r-1}^f). \tag{64}$$

The second and third equality in (Cf.DB) provides a discrete approximation of the constraints $dL(c,\tau)/d\tau > 0$ and $\partial L(c,\tau)/\partial\tau > 0$, respectively. The parameter ϵ is set to be a small positive number, e.g., 10^{-10}, to ensure the derivatives to stay strictly positive.

Finally, we discuss how to formulate the constraint $Q_j(\xi_\ell;\boldsymbol{\tau}^f) = 0$ to facilitate the computation. If kidney type j is dominated by other kidney types for patient ξ_ℓ, then $Q_j(\xi_\ell;\boldsymbol{\tau}^f) = 0$; otherwise, $Q_j(\xi_\ell;\boldsymbol{\tau}^f)$ can be expressed according to Equation (16) using the cutoff values $(\Gamma_j(\xi_\ell;\boldsymbol{\tau}^f))$ associated with $\boldsymbol{\tau}^f$. The cutoff values $(\Gamma_j(\xi_\ell;\boldsymbol{\tau}^f))_{j=1,\dots,J-1}$ can be computed according to the following procedure for given $\boldsymbol{\tau}^f$. First, for all $1 \le j < j' \le J$, we compute variables $\eta_{j,j'}$ as the unique solution to the following equation

$$c(\tau_j^f;\xi_\ell)\phi_j = \eta \int_{\tau_j^f(\xi_\ell)}^{\tau_{j'}^f(\xi_\ell)} \frac{f_{\xi_\ell}(t)}{\bar{F}_{\xi_\ell}(\tau_j^f(\xi_\ell))}(t - \tau_j^f(\xi_\ell))\,dt + \frac{\bar{F}_{\xi_\ell}(\tau_{j'}^f(\xi_\ell))}{\bar{F}_{\xi_\ell}(\tau_j^f(\xi_\ell))}(\eta(\tau_{j'}^f(\xi_\ell) - \tau_j^f(\xi_\ell)) + c(\tau_{j'}^f(\xi_\ell);\xi_\ell)\phi_{j'}). \tag{65}$$

In particular, $\eta_{j,j'}$ stands for the cutoff values at which the patient with initial health score ξ_ℓ is indifferent between accepting a kidney j or turning it down and wait for a kidney of type j'. Note that it could happen that $\eta_{j,j'} < 0$, which suggests that all patients prefer kidney type j' to j; or $\eta_{j,j} > 1$, which suggests that all patients prefer kidney type j to j'. Then starting from $j = 1$, we know for a fixed j, $\min\{\eta_{j,j'} \,|\, j' > j\}$ provides the exact cutoff such that all patients with η smaller that cutoff point prefers kidney type j then any kidney types larger than j'; and all other patients prefer some kidney type greater than j' rather than j. To ensure that Γ_j is non-decreasing in j and within the interval $[0, 1]$, we let

$$\Gamma_j(\xi_\ell; \boldsymbol{\tau}^f) = \min\{1, \max\{\min\{\eta_{j,j'} \,|\, j' > j\}, \Gamma_{j-1}(\xi_\ell; \boldsymbol{\tau}^f)\}\}, \quad \text{for } j = 1, \ldots, J, \tag{66}$$

with $\Gamma_0(\xi_\ell; \boldsymbol{\tau}^f) = 0$ by abuse of notation. Thus, $\Gamma_j(\xi_\ell; \boldsymbol{\tau}^f)$ and therefore $Q_j(\xi_\ell; \boldsymbol{\tau}^f)$ both have an analytical representation, which allow us to compute their sub-gradient with respect to $\boldsymbol{\tau}$ and use first-order methods to solve the policy design problem, e.g., (31) and (33).

Appendix K: Comparison to the Stochastic Setting

We simulate the stochastic waitlist system in which patients and kidneys arrive according to a homogeneous Poisson process, and patients use historical information (e.g., the average score thresholds for each kidney type in the past year) to predict their allocation times and decide whether to accept or reject an offered kidney. All the parameters in stochastic system, including $\rho(\cdot)$, $h(\cdot)$, $c(\cdot;\cdot)$, and $G_\xi(\cdot)$, take the same values as those in the fluid model. We simulate the stochastic waitlist under a matching policy and a healthier-first policy for illustration. Table 3 reports their allocation outcomes in terms of the percentage of patients in each class that transplant each type of kidneys. The reported percentages are averaged over a ten-year period after the waitlist population stops further growing, so these percentages characterize the steady-state allocation outcome. As shown in Table 3, the simulated percentages are all within 5% of those predicted by the fluid model, suggesting that the fluid model has provided accurate predictions. This justifies our model choice.

Appendix L: Sensitivity Analysis

We present a sensitivity analysis to validate the robustness of our results. We change the total patient arrival rate to 641.38, which is 80% of the value we used in Section 5. Keeping all other parameters the same, we use the achievable region to compute the Pareto frontier of the four policies, with the results plot in Figure 6. The plots show similar pattern as in Figure 3, which supports the robustness of our conclusions.

Table 3 Comparison of the Allocation Outcomes for Fluid and Stochastic Models

	Fluid Model	Simulation
Matching	$\begin{pmatrix} 0.070 & 0 & 0 & 0 \\ 0.070 & 0 & 0 & 0 \\ 0.163 & 0 & 0 & 0 \\ 0.003 & 0.181 & 0 & 0 \\ 0 & 0.296 & 0 & 0 \\ 0 & 0 & 0.811 & 0 \\ 0 & 0 & 0.344 & 0.394 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.069 & 0 & 0 & 0 \\ 0.067 & 0 & 0 & 0 \\ 0.160 & 0.001 & 0 & 0 \\ 0.012 & 0.179 & 0 & 0 \\ 0 & 0.292 & 0 & 0.016 \\ 0 & 0 & 0.815 & 0.002 \\ 0 & 0.011 & 0.337 & 0.417 \\ 0 & 0 & 0 & 0.960 \\ 0 & 0 & 0 & 0.982 \\ 0 & 0 & 0 & 0.969 \end{pmatrix}$
Healthier-First	$\begin{pmatrix} 0.002 & 0 & 0 & 0 \\ 0.028 & 0 & 0 & 0 \\ 0.184 & 0 & 0 & 0 \\ 0.050 & 0.180 & 0 & 0 \\ 0 & 0.300 & 0.115 & 0 \\ 0 & 0 & 0.597 & 0 \\ 0 & 0 & 0.406 & 0.495 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.003 & 0 & 0 & 0 \\ 0.028 & 0 & 0 & 0 \\ 0.185 & 0 & 0 & 0 \\ 0.046 & 0.169 & 0.004 & 0 \\ 0 & 0.317 & 0.129 & 0 \\ 0 & 0 & 0.629 & 0 \\ 0 & 0 & 0.400 & 0.475 \\ 0 & 0 & 0 & 0.987 \\ 0 & 0 & 0 & 0.951 \\ 0 & 0 & 0 & 0.971 \end{pmatrix}$

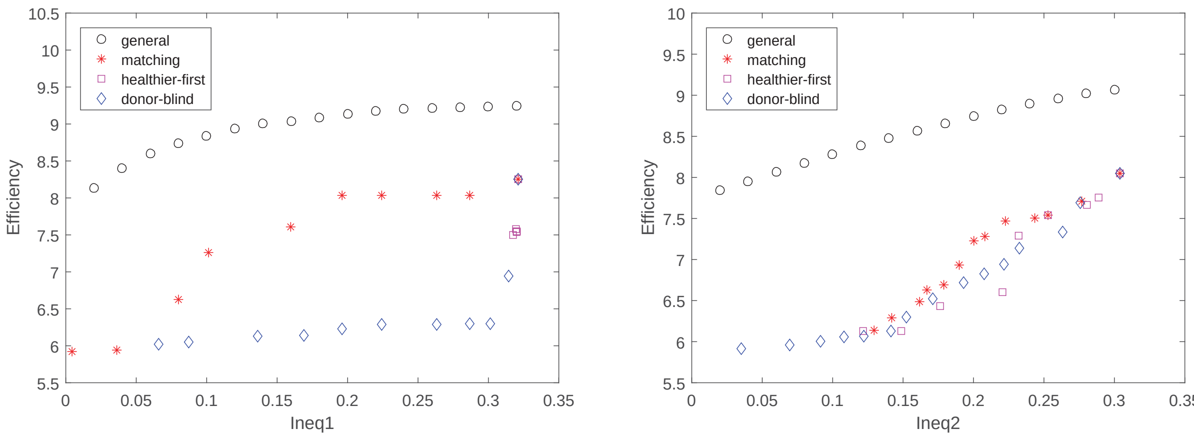

Figure 6 The Efficiency-Equity Pareto Frontier for $\lambda = 641.38$