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Parallel Queues with Discrete-Choice Arrival Pattern: Empirical Evidence and Asymptotic Characterization

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We consider a parallel-queue system in which each queue is served by a dedicated service provider. The arrival process is driven by a discrete choice model, that is, customers observe the queue length for each service provider and choose one to join upon arrival. We assume that a customer's utility is the difference between the service reward and the waiting cost, both of which are heterogeneous. Empirical analysis of the vehicle queues at the U.S.-Canada border-crossing port of entry supports our model setting. We show that with such a choice model, the arrival rate function satisfies certain properties, which allow us to characterizes the fluid and diffusion limit of the queue-length process. In particular, we show that even without the well-used Lipschitz-continuity assumption, the fluid limit process is unique and is attracted to a unique equilibrium. The diffusion limit process is a reflected multi-dimensional Ornstein-Uhlenbeck process centered at that equilibrium. We prove that the stationary distribution of the diffusion limit is a truncated multivariate Gaussian and interchange of limits holds.

Key words: Discrete Choice Model, Nonlinear Complementarity Problem, Fluid and Diffusion Approximation, Reflected Multi-Dimensional Ornstein-Uhlenbeck Process

1. Introduction

The discrete choice model has been widely explored in the literature to model consumer behavior. As a typical scenario studied in the literature, a consumer chooses from an assortment of products with different features and prices to maximize her utility. A customer's net utility of choosing a product is the difference between the reward and cost of obtaining the product. In this paper, we consider an analogue of this discrete-choice model for a service system subject to congestion. Suppose a stochastic service system consists of multiple service providers (SPs) with different features and speeds. Despite the differences, a customer can be served by any one of these SPs, and the service utility depends on the characteristics of both the customer and the SP. We assume that all customers have complete information about the service utility and the expected waiting time for each SP, which allows each customer to calculate the expected net utility of joining each queue. Alternatively, a customer may choose not to join any queue and receives zero utility. If we apply the classical 'discrete choice model' to this service system, then a customer will choose the SP that maximizes her expected utility. Figure 1 provides a graphical illustration of such a system.

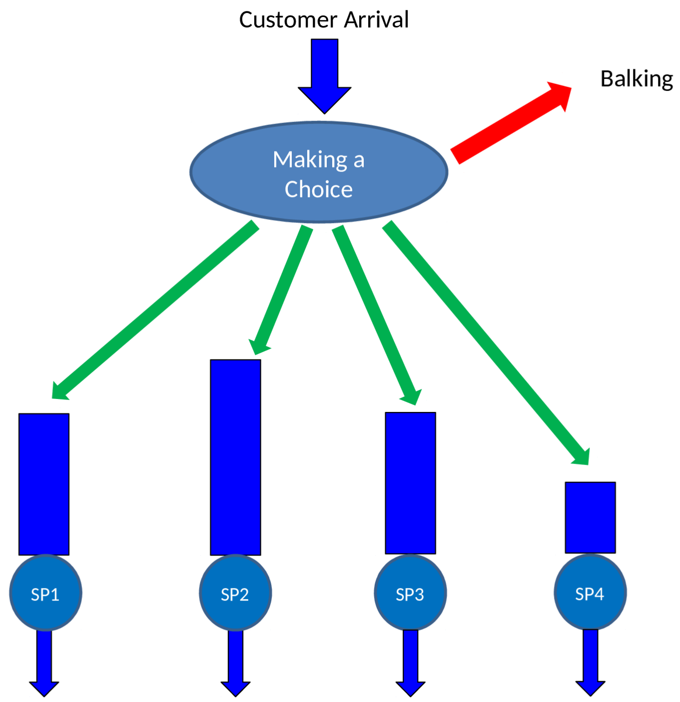


Figure 1 The parallel queue system studied in this paper

We assume that the coefficients in a customer's utility function, which include the service utility and the disutility per unit waiting time, are randomly drawn from the customer population. When real-time waiting time estimates are available to customers, under mild regulation assumptions on the parameter distribution, we can show that the mean arrival rate for each queue is an absolutely continuous function of the waiting time estimates at each different SPs. Furthermore, when the customer's choice is formally modelled, the arrival rate function satisfies the following *waiting-aversion* property: as a queue becomes longer, some customers will be discouraged to join that queue and will instead join other queues or balk. Consequently, the mean arrival rate of a queue decreases with its own length; but is non-decreasing with the lengths of other queues. A formal mathematical description of these properties will be provided in Section 3. For notational brevity, we refer to such properties of the arrival rate function as *choice-driven*. The main objective of this paper is to provide asymptotic characterization of parallel-queues with choice-driven arrivals, or briefly, PQCDA.

Our study is motivated by several practical instances that fit the PQCDA model that are both widely observed and are areas of research in the OR literature. One example is the waitlist for kidney transplantation for patients with end-stage renal disease. Kidneys from deceased donors are allocated to patients who have registered on the transplant list according to a given policy. One allocation policy proposed and tested by Su and Zenios (2006) partitions kidneys into M types by their quality. Arriving patients choose a certain type of kidney and wait in the corresponding queue. Thus, the waitlist virtually consists of M parallel queues, each corresponding to a unique type of service (organs). The stylized models analyzed by Su and Zenios (2006) and Ata et al. (2019) are a simplified version of the PQCDA, by assuming that the patient uses the steady-state queue-lengths to calculate the corresponding waiting times; whereas in reality patients use the real-time queue-lengths. The PQCDA models the latter situation.

The second example is related to an impetus in some health care systems in North America where real time emergency room wait times in specific geographic areas are available online. For example, the web-site `edwaittimes.ca`, shows real time wait times for major hospitals with emergency rooms in the Metro Vancouver area. Patients with preferences on locations and wait times can use this information to choose the hospital they seek care from.

The third example is the international border crossing facilities located between the U.S. and Canada. In the Pacific northwest, there are four border crossing facilities. Almost realtime wait time at each one of these facilities is available. Travellers have preferences for location and the amenities available at each facility and make their choice based on the wait time and the characteristics of each facility. Using a novel data from the Canada-US border crossing in the Pacific Northwest, we validate the key assumptions about the travellers' arrival process.

There is rich literature on queueing systems with customer choice. A number of assumptions about the number of queues (usually a single queue) or congestion information (usually non-real-time) or consumer types (usually single class and homogeneous) or server types (usually homogeneous) have to be made in the literature. However, many of these assumptions may not apply to stochastic service systems in practice. The PQCDA model does not impose any of these restrictive assumptions. Thus, not surprisingly, an exact analysis of PQCDA is challenging. For this reason, we study the queue-length process of PQCDA using fluid and diffusion approximations. Even under such approximations, however, few results are known for parallel queues with general state-dependent arrival rates, e.g., the existence of a system equilibrium, and stationary distribution of the queue length process, etc. See Section 2 for a more detailed literature review. However, we show that these results hold when the arrival rates satisfy the choice-driven properties.

We develop the following approximations for PQCDA. First, under the fluid approximation, we show that the fluid limit process converges to a unique equilibrium which can be characterized as the solution to a nonlinear complementarity problem (NCP). Second, using the diffusion approximation, we show that under the heavy traffic regime, the scaled queue-length process converges to a reflected multi-dimensional Ornstein-Uhlenbeck (RMOU) process, which possesses a unique stationary distribution with closed-form density function (truncated multivariate Gaussian) under certain conditions. We also prove that interchange of limit holds, that is, the stationary distribution of the scaled queue-length process converges to the stationary distribution of the RMOU.

By establishing the above results, we make several important contributions to the related research domain.

1. We propose a fairly general model, i.e., the PQCDA, which captures an important type of customer queue-joining behavior. This type of behavior has been identified in our empirical analysis of the Canada-U.S. border-crossing traffic data, and we believe that it has many other applications.
2. We approximate the transient and stationary behaviors of the queue-length process in PQCDA via fluid and diffusion approximation. In particular, we prove that the fluid limit process converges to a unique equilibrium state, and that the diffusion limit process is an RMOU, whose covariance matrix depends on the degree of substitutability between different SPs as well as the customers' delay sensitivity. These characterizations provide system managers with qualitative insights to the long-term behavior of PQCDA.
3. We propose an algorithm to compute the equilibrium state of the fluid limit process, and derive the closed-form stationary distribution for the diffusion limit process. These results allow system manager to calculate service-level related measures, which are useful for both performance evaluation as well as capacity planning for the service system. These results also facilitate the evaluation of other performance measures such as the value of real time information, individual customer's benefits, and the social welfare for large systems, for which the asymptotic analysis provides reasonably close approximations.

4. The choice-driven property allows us to establish the following technical results in lieu of the Lipschitz continuity assumption: uniqueness of the fluid limit process, convergence of the original stochastic process to the fluid limit and diffusion limit, and interchange of limits. We show that these results may not hold in parallel queues with general, non-Lipschitz arrival rates; but they hold when the arrival rates are non-Lipschitz but have the choice-driven properties. We thus provide a new proof technique for the above results that does not rely on the Lipschitz assumption as the classical methods (e.g., (Mandelbaum et al., 1998a,b)) do. The technical results may be of independent interest to the applied probability society.

The rest of the paper is organized as follows. Section 2 presents a literature review. Section 3 provides a formal definition of the PQCDA model. Section 4 presents an empirical study of the border-crossing traffic data. Section 5 introduces some notations and preliminary results that facilitate the subsequent asymptotic analysis. In Section 6 and Section 7, we derive the fluid and diffusion approximations for the queue-lengths process in PQCDA, respectively. Section 8 extends the results to the case with customer reneging. Finally, Section 9 concludes with a summary of the paper and a discussion of future research.

2. Literature Review

The first stream of papers focus on modeling and analyzing the effect of arriving customers' queue-joining behavior in various queueing systems. These models are classified in Figure 2. As shown in Figure 2, first, there are two general classes of works in this area classified according to "information level" (IL) with O for observable and U for unobservable queues. Each class is categorized into six types of models according to "number of queues" (NQ) with M for multiple queues and S for single queue, "customer class" (CC) with H for homogeneous and T for heterogeneous customers, and "server type" (ST) with I for identical and D for different servers. Thus, each type of model can be denoted by the notation with four letters separated by backslash (to distinguish from the forward slash used for Kendall notation). For example, our model can be denoted as $O\backslash M\backslash T\backslash D$ meaning a system with observable multiple queues, heterogeneous customers, and different servers. Customers are different in delay sensitivity and service value, but have the same service rate at the same server, while servers are different in service value and service rate. Note that for each node in Figure 2, the left branch is the special case of the right branch. In reviewing the literature, it will be clear that the model we treat here is a more general version of the observable queue setting with customer choice, the one which has been less studied in the literature. In the literature review on the models in the above classification, we mainly focus on those papers that are directly related to our model. A more exhaustive reference can be found in a monograph by Hassin et al. (2006).

Some of the early models of the $O\backslash S\backslash H\backslash I$ type are by Naor (1969) and Leeman (1964) who investigated homogeneous customers' decisions on whether to join a queue for service. When the queue is observable, they showed that in equilibrium, a pure threshold strategy (i.e., joining the queue when the queue length is below a threshold) maximizes consumer surplus. However, this equilibrium solution is sub-optimal with respect to the social welfare. The socially optimal solution is reached by introducing an admission cost (toll) in addition to the waiting cost as shown in Stidham Jr (1978). Hassin (1986) found that in a last-come-first-serve queue with customer abandonment, the differences between Pareto optimal and social optimal equilibria due to possible customers' negative externality does not arise. Larsen (1998) generalized Noar's model to the one with heterogeneous customers who differ in service value. In contrast, Edelson and Hilderbrand (1975) and Frutos and Gallego (1999) studied the heterogeneous customer model where two classes of customers differ in their marginal waiting cost. The above models belong to $O\backslash S\backslash T\backslash I$ type. When there are multiple parallel observable queues, homogeneous customers, and identical servers (i.e., the $O\backslash M\backslash H\backslash I$ type model), the system generally does not have an equilibrium as indicated in Hassin et al. (2006), except for some special models (e.g. Hassin (2009)). For this reason, the $O\backslash M\backslash H\backslash I$ type models are studied under a weaker notion of equilibrium such as the "ϵ-equilibrium". An example of $O\backslash M\backslash H\backslash D$ type model was considered in Li and Lee (1994). They considered a setting with two queues with heterogeneous servers and homogeneous customers where balking is not allowed but jockeying is permitted. The most general case is the $O\backslash M\backslash T\backslash D$ type model, which is the far right branch in observable queue class in Figure 1. The PQCDA studied in this paper falls into this category as we assume customers have different sensitivity with delay and heterogeneous preferences among SPs. Related studies in this category focus on the case where customers receive delayed information about waiting time estimates; see Pender et al. (2020) and Dong et al. (2019). There are several fundamental differences between our work and these two papers. Two important ones are: (1) our paper considers a more general customer choice model which requires different analytical methods and (2) the steady-state characterizations derived for our model may not hold when information is delayed.

The first study on the simplest unobservable queue case or $U\backslash S\backslash H\backslash I$ type was done by Edelson and Hilderbrand (1975) and Chen and Frank (2004). Two extensions followed the basic unobservable queue model. Littlechild (1974) considered an M/M/1 queue with customers of heterogeneous service values which falls under the $U\backslash S\backslash T\backslash I$ type. Later, Mendelson (1985) extended the model to a more general GI/G/s setting. Luski (1976) generalized the model in Edelson and Hilderbrand (1975) to a two-queue system which belongs to the $U\backslash M\backslash H\backslash I$ type and studied the equilibrium pricing strategies. Recently, Hua et al. (2014) studied two-tier service systems with either identical or multi-class customers which are examples of $U\backslash M\backslash T\backslash I$ type or $U\backslash M\backslash T\backslash D$ type but they focused on the two queue case only. Thus most models in the unobservable queue class have been studied in the literature and are relatively well understood. Other queueing models involving strategic behavior of customers or servers include Adiri and Yechiali (1974); Maglaras et al. (2016); Afèche and Ata (2013); Ward and Armony (2013); Ibrahim et al. (2016); Dong et al. (2015); Gupta and Zhang (2014).

The second stream of related research is the one on fluid and diffusion approximations for service systems with multiple queues. In the models in this stream, the system state is usually represented by a vector, with each component representing the length of a queue. There is a rich literature that models this type of systems as multi-dimensional diffusion processes. The closest model to the PQCDA is the state-dependent queueing network studied in Haddad and Mazumdar (2012); Lee and Puhalskii (2015); Leite and Fragoso (2008); Mandelbaum et al. (1998a,b); Yamada (1995), with some important differences. Compared to a general state-dependent queueing network model, the choice-driven property allows us to derive several characterizations for the fluid and diffusion limit processes (e.g., the fluid limit process converges to a unique equilibrium point, the diffusion limit process is an RMOU process, whose steady-state distribution admits a closed-form characterization). Those characteristics are otherwise not valid in a general state-dependent queueing network. Furthermore, we show that the choice-driven property can substitute for the Lipschitz property in the proofs of the above results in Mandelbaum et al. (1998b). Other papers on state-dependent queues investigated the case when the service speed depends on the workload in the buffer, e.g., (Abouee-Mehrizi and Baron, 2016; Delasay et al., 2016; Dong et al., 2015).

There are several queueing models in which the fluid and diffusion limits exhibit similar ergodic properties. A well-known example is a queueing network with constant arrival rates and constant or state-dependent routing matrix; see Harrison and Reiman (1981) and Reiman (1984). In these models, the fluid limit process has a unique equilibrium $\mathbf{0}$, owing to the negative drifts and the non-negative constraint enforced by the reflecting barrier. Consequently, the diffusion limit process in those models is a multi-dimensional Brownian motion with a reflection barrier at $\mathbf{0}$. These characterizations differ from the PQCDA, in which the equilibrium state of the fluid process results from the choice-driven property, and the diffusion limit is thus an RMOU rather than a reflected Brownian motion. Another related model is an overloaded queueing network where customers in

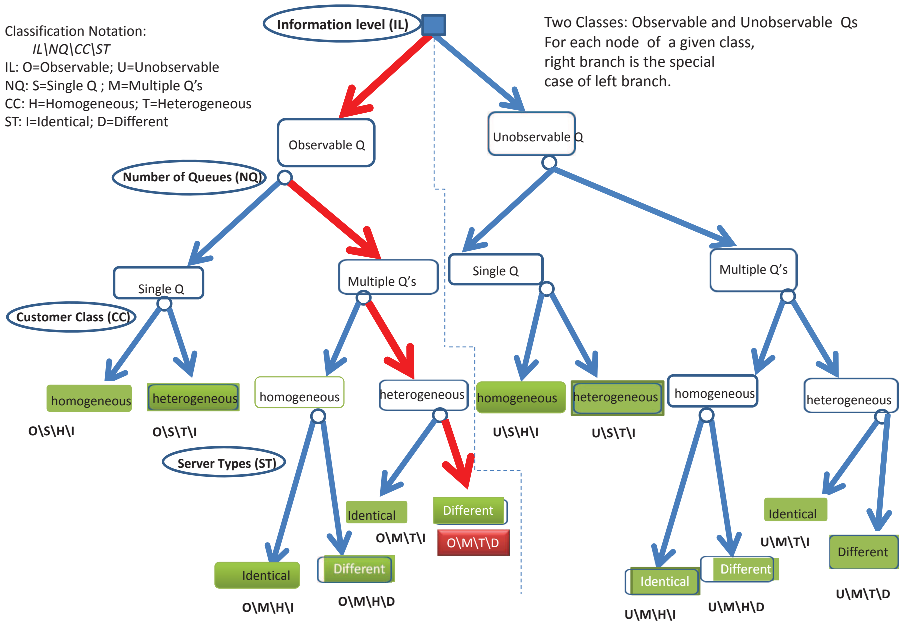


Figure 2 Classification of Queueing Models with Customer Choice.

each queue renege after an exponentially distributed time. For such a model, Reed and Ward (2004) showed that the fluid limit has a non-zero equilibrium and the diffusion limit process is a non-reflected multi-dimensional O-U process. Other similar models include a service system with differentiated service levels in Maglaras and Zeevi (2004), or with heterogeneous customer types in Harrison and Zeevi (2004). In all these models, the drift is a linear function of the system state; whereas our model allows the drift function to be nonlinear and possibly non-smooth. Therefore, to adapt the existing methods to our model, we need to show that the original process can be approximated by a diffusion process with a linear drift when it is close to the equilibrium.

3. The PQCDA Model

3.1. Discrete Choice Model

We consider a system with J parallel heterogeneous service providers, indexed by $j = 1, 2, \ldots, J$. We assume that the customers' queue-joining behavior follows the classical discrete choice model (e.g. Train (1986)). We show that the resulting arrival rates satisfy the choice-driven properties that will be defined later in this section. Formally, for a customer of type ξ, the information available to that customer includes the service utility at the j^{th} SP, $u_{\xi,j}$, the customer's waiting cost per unit time, c_ξ, and the system state. The system state can be described by a J-dimensional vector of waiting time estimates for the customer to join each queue right before time t, that is, $\boldsymbol{\tau}(t-) := (\tau_j(t-))_{j=1,\ldots,J}$, where $\boldsymbol{\tau}(t-)$ denotes the left-limit of $\boldsymbol{\tau}(\cdot)$ at time t. We assume that there are uncountably many different customer types ξ. Note that both the service utility $u_{\xi,j}$ and the waiting cost c_ξ vary by customer type ξ, and can be regarded as random variables that follow a fixed probability distribution. Given waiting time estimates $\boldsymbol{\tau}(t-)$, a customer indexed by ξ can compute her expected utility $U_{\xi,j}$ of joining the j-th queue at time t as follows,

$$U_{\xi,j} = \begin{cases} u_{\xi,j} - c_\xi \tau_j(t-), & \text{if } j \neq 0 \text{ (joining)} \\ 0 & \text{if } j = 0 \text{ (balking)} \end{cases} \tag{1}$$

With the utility function defined in (1), the choice problem for a customer indexed by ξ can be formulated as

$$\arg\max_{0 \le j \le J} \{0, U_{\xi,1}, U_{\xi,2}, \ldots, U_{\xi,J}\}, \tag{2}$$

where the utility of balking is assumed to be zero without loss of generality. For example, suppose an arrived customer sees two queues with waiting time estimates $\tau_1(t) = 1$ and $\tau_2(t) = 2$. If the parameters of a customer are $u_{\xi,1} = 0$, $u_{\xi,2} = 3$, and $c_\xi = 1$, then his utility of joining queue 1 and 2 are $U_{\xi,1} = -1$ and $U_{\xi,2} = 1$, respectively, in which case he will join queue 2. If we change the value of c_ξ from 1 to 2, then his utility will be $U_{\xi,1} = -2$ and $U_{\xi,2} = -1$, in which case he will choose to balk and receives a utility $U_{\xi,0} = 0$.

Since the parameters $\boldsymbol{u}_\xi := (u_{\xi,j})_{j=1,\ldots,J}$ and c_ξ have a fixed joint distribution, we can compute the probability for a randomly drawn arrived customer to choose a queue $j = 0, 1, \ldots, J$, where queue 0 corresponds to balking by slightly abuse of notation. The choice probabilities have the following expressions,

$$\begin{aligned} p_0(\boldsymbol{\tau}(t-)) &= \Pr(0 > u_{\xi,k} - c_\xi \tau_k(t-)\ ,\ k = 1, \ldots, J) \\ p_j(\boldsymbol{\tau}(t-)) &= \Pr(u_{\xi,j} - c_\xi \tau_j(t-) > 0 \text{ and } u_{\xi,j} - c_\xi \tau_j(t-) > u_{\xi,k} - c_\xi \tau_k(t-),\ k = 1, \ldots, J,\ k \neq j). \end{aligned} \tag{3}$$

As will be discussed later, we assume that $(\boldsymbol{u_\xi}, c_\xi)$ has a continuous distribution and thus a tie happens with zero probability.

Next, we introduce a few assumptions on the distribution of $(\boldsymbol{u}_\xi, c_\xi)$. These assumptions are minimal and are able to accommodate a wide range of applications. Under these assumptions, we prove certain desirable properties of the arrival rate function which in turn facilitate the asymptotic characterization of the PQCDA. Later, we will show that our choice model subsumes several well known models such as the conditional logit model and the mixed logit model. For the sake of brevity, we will omit the subscript ξ and denote the random parameters as u_j and c when there is no ambiguity.

Assumption 1 *(Waiting Aversion)* $c > 0$ *a.e.*

Remark 1 *A main feature of the choice model studied in this paper is that customers have real time queue-length information and are waiting averse. As a consequence, when one queue becomes longer, we expect a larger proportion of customers to join the other queues or to balk. We refer to such behavior as choice-driven property and a formal definition will follow later.*

Define

$$\mathcal{K}^J := \{\boldsymbol{u} \in \mathbb{R}_+^J \mid u_i = u_j \text{ for some } i \neq j\}, \text{ for all } J \ge 2. \tag{4}$$

Assumption 2 *(Continuous Effect)* $(\boldsymbol{u}, c)$ *has an absolute continuous cumulative distribution function (cdf) and its joint probability density function (pdf)* $f(\boldsymbol{u}, c)$ *is positive and finite almost everywhere on the domain* $\mathbb{R}^J \otimes \mathbb{R}_+$ *except when* $\boldsymbol{u} \in \mathcal{K}^J$ *or* $c = 0$, $f(\boldsymbol{u}, c)$ *can be infinitely large.*

Remark 2 *Intuitively, the above assumption requires the parameters* $(\boldsymbol{u}, c)$ *to spread continuously over its domain. As a result, any change of queue length will affect the choices of a small but positive proportion of customers. Therefore, any queue-length change always has a non-zero* *but continuous impact on the mean arrival rate for each queue. Our subsequent analysis of the PQCDA is built on this property. Without continuity of the cdf, we will have significantly different queue-joining behavior and system dynamics. For example, if all customers have the same parameter* $(\boldsymbol{u}, c)$*, then customers will join a queue if and only if its length is below a fixed threshold (e.g., Hassin et al. (2006)). If all customers have the same service value, i.e.,* $u_1 = u_j$ *for all* j*, then we have a join-the-shortest-queue (JSQ) model. The asymptotic analysis for JSQ takes a different approach from that for the PQCDA; see (Eschenfeldt and Gamarnik, 2018; Cao et al., 2019).*

Remark 3 *Although we have to assume that* $(\boldsymbol{u}, c)$ *has a continuous distribution without any point mass, we do not want to overlook the existence of two special types of customers. The first type of customers are insensitive with waiting and have* $c = 0$*; the second type of customers are indifferent between several different SPs, corresponding to the case of* $\boldsymbol{u} \in \mathcal{K}^J$*. To approximate the potential existence of point mass at those points, we allow the pdf function* $f(\cdot)$ *to take infinitely large values when* $c = 0$ *or when* $\boldsymbol{u} \in \mathcal{K}^J$*. As a result, the cdf function will be continuous but may not have a finite derivative at those points.*

The above general formulation subsumes several well known choice models. If we assume that

$$u_{\xi,j} = v_{\xi,j} + \epsilon_{\xi,j}. \tag{5}$$

where $\epsilon_{\xi,j}$ has an i.i.d. standard type-1 extreme value distribution. Then we get a *mixed logit model* (Train, 2009) and the choice probability is given by

$$\begin{aligned} p_0(\boldsymbol{\tau}) &= E_\xi[\tfrac{1}{1+\sum_{k=1}^J \exp(v_{\xi,k} - c_\xi \tau_k)}], \\ p_j(\boldsymbol{\tau}) &= E_\xi[\tfrac{\exp(v_{\xi,j} - c_\xi \tau_j)}{1+\sum_{k=1}^J \exp(v_{\xi,k} - c_\xi \tau_k)}], \text{ for } j = 1, \ldots, J. \end{aligned} \tag{6}$$

If we further assume that the coefficients are homogeneous among the population, that is, $v_{\xi,j} \equiv v_j$ and $c_\xi \equiv c > 0$ for all ξ, then we have the conditional logit model McFadden et al. (1973). The choice probability is given by

$$\begin{aligned} p_0(\boldsymbol{\tau}) &= \tfrac{1}{1+\sum_{k=1}^J \exp(v_k - c\tau_k)}, \\ p_j(\boldsymbol{\tau}) &= \tfrac{\exp(v_j - c\tau_j)}{1+\sum_{k=1}^J \exp(v_k - c\tau_k)}, \text{ for } j = 1, \ldots, J. \end{aligned} \tag{7}$$

Similarly, we will get a probit model by assuming ϵ to follow an i.i.d. standard normal distribution.

3.2. Arrival Process

We next characterize the arrival process under the discrete choice model. Formally, we assume that the service times at server j are i.i.d. random variables with a finite mean $1/\mu_j$. We use the vector notation $\boldsymbol{\mu} := \{\mu_j\}_{j=1,\ldots,J}$. Customers arrive at the system according to a time-homogeneous Poisson process with a constant rate 1. When a customer arrives at the system, he decides whether to join any one of the J queues or balk. After a customer joins a queue, abandonments and switching between queues are not allowed (Though an extension of the model with exponential customer reneging time is doable and discussed in Section 8). The service discipline is First-Come-First-Served (FCFS) at each queue. A customer leaves the system permanently after service completion.

We describe the system state at time t using a queue-length vector $\boldsymbol{X}(t) := (X_j(t))_{j=1,\ldots,J}$, where $X_j(t)$ denotes the number of customers in queue j including the one currently in service. In most practical applications of PQCDA, the remaining service time of the customer at the head of line cannot be observed by either the customer or the system manager. Therefore, we assume that the customers or the system manager will simply use the average service time of a new job to estimate that remaining service time. This approximation is typically accurate because the queue length in many realistic applications of the PQCDA are usually much larger than one. Using this approximation, the waiting time estimator $\tau_j(\xi)$ has the following expression:

$$\tau_j(t) = \frac{X_j(t)}{\mu_j}. \tag{8}$$

In the rest of the paper, we refer to $\tau_j(t)$ as the waiting time estimate or the delay estimate.

Recall that we use $p_j(\boldsymbol{\tau})$ $(j = 0, 1, \ldots, J)$ to denote the probability for a randomly arriving customer to choose queue j, which is assumed to be independent of the arrival sequence. Since the aggregate arrival rate is one, the mean arrival rate for queue j is exactly $p_j(\boldsymbol{\tau})$. We thus refer to $p_j(\cdot)$ as the *arrival rate function.* We assume that $p_j(\cdot)$ satisfies the following *stability condition*,

$$\exists\ K > 0, \text{ such that } p_j(\boldsymbol{\tau}) < \mu_j \text{ for all } \boldsymbol{\tau} \in \mathbb{R}_+^J \text{ wit } \tau_j \geq K. \tag{9}$$

The above condition guarantees that whenever a queue is sufficiently long, the state-dependent arrival rate is strictly capped by the service capacity, so the queue length will be bounded. Equation (9) can be considered as the "state-dependent" version of the well-known stability condition "$\lambda < \mu$" in a single queue.

Let $\boldsymbol{\Lambda}(\boldsymbol{\tau}) := (p_j(\boldsymbol{\tau}))_{j=1,\ldots,J}$ denote the vector of the state-dependent arrival rates. Let $\boldsymbol{R}(\boldsymbol{\tau}) := (\frac{\partial p_i(\tau)}{\partial \boldsymbol{\tau}_j})_{i,j=1,\ldots,J}$ denote the Jacobian matrix of $\boldsymbol{\Lambda}(\boldsymbol{\tau})$ when it exists. We next provide a formal definition of the choice-driven property.

Definition 1 *The function $\boldsymbol{\Lambda}(\boldsymbol{\tau}) := (p_j(\boldsymbol{\tau}))_{j=1,\ldots,J}$ is said to satisfy the choice-driven (CD) property if it is absolutely continuous in $\boldsymbol{\tau}$, and its Jacobean matrix $\boldsymbol{R}(\boldsymbol{\tau})$ is continuous everywhere*[1] *and satisfies the following properties for almost every $\boldsymbol{\tau} := (\tau_j)$:*

1. *(CD-a) Non-Negative Off-Diagonals:*

$$p_j(\boldsymbol{\tau}) \text{ is non-decreasing in } \tau_k \text{ for } j = 1, \ldots, J \text{ and } k \neq j. \tag{10}$$

Or equivalently, its Jacobean $\boldsymbol{R}(\boldsymbol{\tau})$ has non-negative off-diagonal entries.

2. *(CD-b) Negative Diagonals:*

$$p_j(\boldsymbol{\tau}) \text{ is strictly decreasing in } \tau_j \text{ for } j = 1, \ldots, J. \tag{11}$$

Or equivalently, its Jacobean $\boldsymbol{R}(\boldsymbol{\tau})$ has negative diagonal entries.

3. *(CD-c) Strict Row and Column Diagonal Dominance:*

$$p_j(\boldsymbol{\tau} + t\boldsymbol{e}) < p_j(\boldsymbol{\tau}) \text{ for } j = 1, \ldots, J,\ t > 0, \tag{12}$$

where $\boldsymbol{e}$ denotes an all-one vector. Or equivalently, $\boldsymbol{R}$ has negative row sums.

$$\sum_{k=1}^{J} p_k(\boldsymbol{\tau} + te_j) < \sum_{k=1}^{J} p_k(\boldsymbol{\tau}) \text{ for } j = 1, \ldots, J,\ t > 0, \tag{13}$$

where e_j denotes a vector with its j^{th} entry equal to one and all other entries equal to zero. Or equivalently, $\boldsymbol{R}(\boldsymbol{\tau})$ has negative column sums.

Remark 4 *The (CD) property implies that the Jacobean matrix $\boldsymbol{R}(\cdot)$ is non-symmetric negative definite a.e. For a negative non-symmetric definite matrix, all of its eigenvalues have negative real parts (see e.g. Plemmons and Berman (1979)).*

[1] We allow the partial derivative $\partial p_j(\boldsymbol{\tau})/\partial \tau_i = +\infty\,(-\infty)$ at some point $\boldsymbol{\tau}$. Then continuity at $\boldsymbol{\tau}$ means $\lim_{n\to\infty} \partial p_j(\boldsymbol{\tau}^n)/\partial \tau_i \to +\infty\,(-\infty)$ for any sequence $\boldsymbol{\tau}^n \to \boldsymbol{\tau}$.

Remark 5 *Because $\boldsymbol{R}$ is not bounded, the arrival rate function $\boldsymbol{\Lambda}(\boldsymbol{\tau})$ is not Lipschitz continuous. In fact, it is even not locally Lipschitz continuous because $\boldsymbol{R}$ may contain infinite entries when $\boldsymbol{\tau} \in K^J$. We provide an example in the end of Appendix A.*

We next provide some intuition towards the above properties for the arrival rate function. Note that $\tau_j(t)$ is proportional to the queue length. Thus, a larger $\tau_j(t)$ corresponds to a longer queue. Property (CD-a) stands for weak gross substitutability (WGS) across different SPs – the arrival rate tends to increase when other queues become longer. Property (CD-b) means that the arrival rate of a queue decreases when it becomes longer. To interpret Property (CD-c), i.e., Conditions (12) and (13), consider a scenario when the estimated waiting times in all queues have increased by the same amount, then the difference in the expected waiting times across different queues will keep the same. As a result, a customer's preference order between any two queues will not be altered. However, the increased queue lengths lead more customers to balk, so each queue ends up with a smaller arrival rate. This gives strict row diagonal dominance. Also, when one queue becomes longer, it may push some customers to other queues, but may also push some other customers to balk. So the total arrival rate for all queues has to decrease. This gives the column diagonal dominance.

The next proposition shows that the discrete choice model described in Section 3 leads to the CD properties (i.e., (CD-a), (CD-b) and (CD-c)) of the arrival rate function. In fact, we can prove an even stronger property of the arrival rate function – its Jacobean matrix must be symmetric. However, symmetry is only needed to allow the stationary distribution of the diffusion limit process to have a closed form. For the other asymptotic results presented in this paper, it suffices to assume that the arrival rate function $\boldsymbol{\Lambda}(\boldsymbol{\tau}) := (p_j(\boldsymbol{\tau}))_{j=1,\ldots,J}$ satisfies the CD property as well as the stability condition (9).

Proposition 1 *The arrival-rate function given by* (3) *satisfies Properties (CD-a), (CD-b), and (CD-c) as well as the stability condition* (9)*. Moreover, its Jacobean matrix is symmetric almost everywhere.*

The proof of Proposition 1 is provided in Section A.

4. Empirical Evidence

To validate the customer choice model presented in Section 3, we introduce a real life parallel-queue system and investigate the customer choice behavior using real data. We consider automobile queues at the two U.S.-Canada border-crossing ports of entries at the west coast, i.e., Peace Arch and Pacific crossings. The two ports are located within 2 miles of each other and an automobile can cross the border via either port by choosing the corresponding exit to leave the highway. Figure 3 visualizes the geographic locations of the two ports.

To cross the border, every vehicle needs to be screened by an officer at an inspection booth. This process takes a few minutes and creates a bottleneck or a queue for the border-crossing traffic. There are a maximum of eight booths at each port of entry and the number of open booths varies across a day. Since these booths are located next to each other, a vehicle can choose one of the open booths to cross the border. Thus, all vehicles at the same port of entry are in a pooled queue, regardless which booth they actually go through. However, the vehicles at one port of entry cannot switch to the other, so vehicles at the two ports of entry form two separated parallel queues.

The up-to-date waiting time estimates for crossing the two ports are disclosed to travellers on the message boards on the highway (Interstate-5) near the exits to the two crossings. Travellers can also learn about the latest waiting time estimates from in-vehicle radio, which is broadcasted every 10 minutes or less. Travellers can then cross the border through either Peace Arch or Pacific. Thus, the vehicle queues at the two crossings can be modeled as two parallel queues with arrivals from the

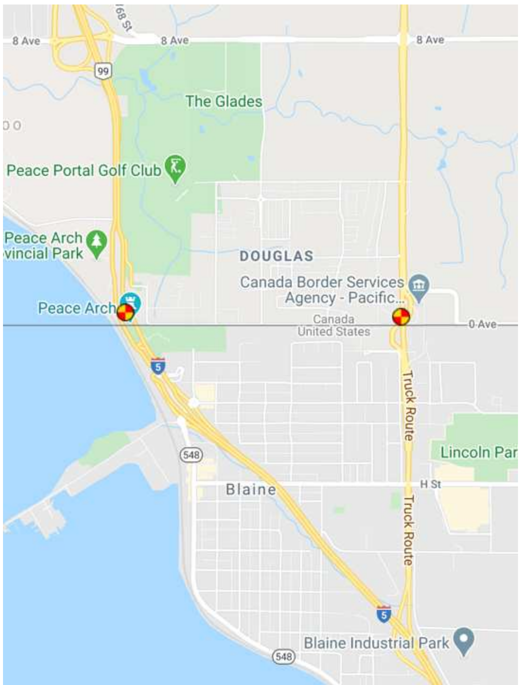


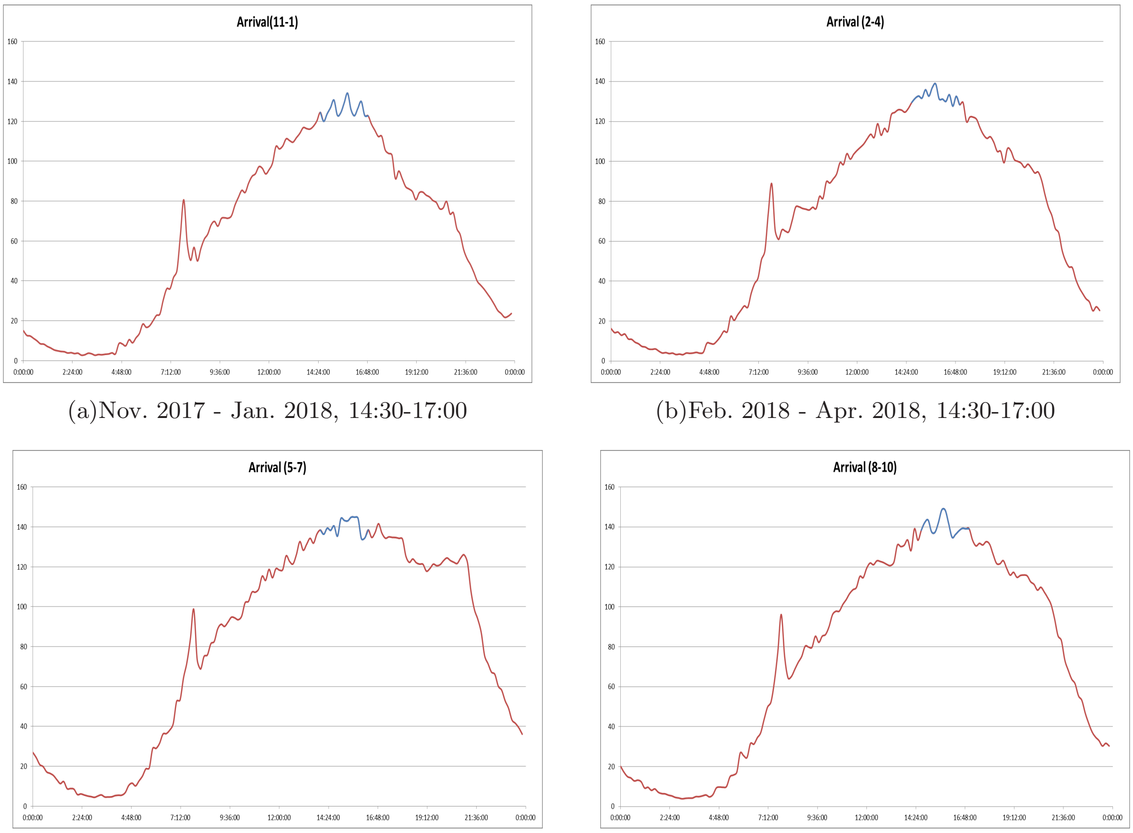
Figure 3 The Peace Arch and Pacific Border-Crossings

highway. We next analyze the historical border-crossing traffic data and show that the travellers are aware of the waiting time estimates and are waiting-averse, which validates Assumption 1.

Our data is collected from the public website (WCOG, 2019). It records the number of arrivals in five minute intervals at each port of entry, denoted by $a_{pe}(t)$ and $a_{pa}(t)$, and waiting-time (delay) estimates at the beginning of every five-minute interval, denoted by $\tau_{pe}(t)$ and $\tau_{pa}(t)$. Here $t = 1, 2, \ldots$ denotes the index of each five-minute interval. Commercial trucks and vehicles with a special dedicated fast lane such as NEXUS, go through separated lanes and are not included in this tally. Anecdotal evidence suggests that some vehicles indeed balk upon observing a long queue at the ports. However, the exact number of balked vehicles cannot be tracked because a vehicle can balk anywhere on its way to the crossing.

Our empirical analysis is based on the northbound border-crossing traffic data in a one-year study period from February 2018 to January 2019. To control the potential seasonal effect, we divide the study period into four seasons: Feb-Apr, May-July, August-October, and November-January. Months with similar intra-day arrival patterns are grouped into the same season. To control the day-of-week effect, we only use traffic data on Tuesday, Wednesday, and Thursday, because the arrival patterns in these days are very similar (Yu et al., 2016). Figure 4 plots the average total arrival rates $a_{pe}(t) + a_{pa}(t)$ for the two ports of entry on Tuesday/Wednesday/Thursday (T/W/T) in each season. Since travellers may pay more attention to the waiting time estimates when there is a substantial delay, we focus on traffics during the peak hours. To that end, we select a fixed 2.5-hours time window among days in the same season, during which the arrival rate reaches a plateau. See Figure 4 for the selection of the peak hours.

After a traveller learns about the waiting-time estimates either from the message board on highway or from the radio, it typically takes him less than five minutes till his vehicle joins the queue at a port of entry and is counted as an arrival. Thus, when predicting the choice probability at the beginning of the t^{th} five-minute slot, we should use the waiting time estimates at the beginning of the $(t - \Delta)^{th}$ slot. In our numerical experiments, as a robustness check we have tested $\Delta = 0, 1, 2$ to capture the possible time lags of 0, 5, and 10 minutes, respectively.



(a)Nov. 2017 - Jan. 2018, 14:30-17:00

(b)Feb. 2018 - Apr. 2018, 14:30-17:00

(c)May. 2018 - Jul. 2018, 14:00-16:30

(d)Aug. 2018 - Nov. 2018, 14:40-17:10

Figure 4 Plots of average total arrival rates on T/W/T in each season, with the peak hours marked in blue.

We want to study the effect of waiting time on travellers' queue-joining behavior. A simple model – the conditional logit model – is sufficient to serve that purpose. More sophisticated methods, such as a mixed logit model, might lead to better goodness-of-fitness, but the conclusions are likely similar. By the IIA (independence of irrespective alternative) property of the conditional logit model, the probability for a passenger to choose Peace Arch instead of Pacific, conditional on that the passenger would not balk, can be calculated as follows,

$$\begin{aligned}\frac{p_{pe}(t)}{p_{pe}(t)+p_{pa}(t)} &= \frac{\exp(v_{pe}-c\tau_{pe}(t-\Delta))}{\exp(v_{pe}-c\tau_{pe}(t-\Delta))+\exp(v_{pa}-c\tau_{pa}(t-\Delta))}\\ &= \frac{1}{1+\exp((v_{pa}-v_{pe})-c(\tau_{pa}(t-\Delta)-\tau_{pe}(t-\Delta)))}.\end{aligned} \tag{14}$$

where $p_{pe}(t)$ and $p_{pa}(t)$ denote the proportion of travellers who choose Peace Arch and Pacific crossing at time t, respectively, v_{pe} and v_{pa} denote the expected service utility, excluding the waiting cost, at Pacific and Peace Arch, respectively, and c denotes the waiting cost per minute. Although we do not have data on the number of balking vehicles, we can derive the nonlinear least square estimator for $\hat{v}_{pa}-\hat{v}_{pe}$ and $\hat{c}$ as

$$(\hat{v}_{pa}-\hat{v}_{pe},\hat{c}) := \arg\min\left(\frac{1}{1+\exp((v_{pa}-v_{pe})-c(\tau_{pa}(t-\Delta)-\tau_{pe}(t-\Delta)))}-\frac{a_{pe}(t)}{a_{pe}(t)+a_{pa}(t)}\right)^2. \tag{15}$$

The estimation values are summarized in Table 1. For all the four seasons and time lags $\Delta = 0, 1, 2$, the estimator of waiting cost $\hat{c}$ is consistently positive at a 0.001 significance level. That provides strong evidence that travellers have paid attention to the waiting time estimates and tried to avoid longer queues during the peak hours. This verifies that Assumption 1 holds for this border crossing system. We also find that travellers' preference, excluding waiting cost effect, changes between Peace Arch and Pacific differ from season to season. From August till January, the coefficient estimator $\hat{v}_{pa} - \hat{v}_{pe}$ stays negative at a 0.001 significance level, suggesting that more travellers prefer Peace Arch to Pacific during those months. However, from February till May, Pacific becomes the preferred crossing. From June till August, the two ports are equally preferred.

Table 1 Estimation Results

		Coefficients (Standard Error)			Odds ratio		
Time Lag		0 min	5 min	10 min	0 min	5 min	10 min
Nov-Jan	$\hat{v}_{pa} - \hat{v}_{pe}$	-0.130***	-0.121 ***	-0.119***	0.878	0.886	0.888
		(0.010)	(0.010)	(0.011)			
	$\hat{c}$	0.004***	0.006***	0.006***	1.004	1.006	1.006
		(0.001)	(0.001)	(0.001)			
Feb-Apr	$\hat{v}_{pa} - \hat{v}_{pe}$	0.027*	0.034 **	0.035**	1.027	1.035	1.036
		(0.010)	(0.011)	(0.011)			
	$\hat{c}$	0.014***	0.016***	0.017***	1.014	1.016	1.017
		(0.001)	(0.001)	(0.001)			
May-Jul	$\hat{v}_{pa} - \hat{v}_{pe}$	-0.004	0.007	0.015	0.996	1.007	1.015
		(0.010)	(0.010)	(0.011)			
	$\hat{c}$	0.009***	0.011***	0.013***	1.009	1.011	1.013
		(0.0007)	(0.0007)	(0.0007)			
Aug-Oct	$\hat{v}_{pa} - \hat{v}_{pe}$	-0.231***	-0.218 ***	-0.213***	0.793	0.804	0.808
		(0.010)	(0.010)	(0.011)			
	$\hat{c}$	0.003***	0.005***	0.006***	1.003	1.005	1.006
		(0.001)	(0.001)	(0.001)			

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

We cannot fully validate Assumption 2, which is a regularity condition and states that the parameters of travellers have a continuous and positive distribution everywhere. Our analysis of border-crossing traffic data, however, supports that customers have heterogeneous service utility at each port of entry, i.e., $u_{\xi,j}$ differs for different ξ. Because if all customers have the same service utility, which means that $u_{\xi,j} \equiv v_j$ and the random error $\epsilon_{\xi,j} \equiv 0$ for all ξ. Then the existing results in the literature (e.g., (Hassin et al., 2006)) implies that the mean arrival rate to one queue is either a positive constant or zero, depending on whether the difference in queue lengths is below or above a threshold. Nevertheless, empirical data analysis shows that the arrival rate changes continuously with the queue lengths. Thus, it is appropriate to assume that $u_{\xi,j}$ follows a distribution within the population.

5. Notations and Preliminaries

This section introduces some notations and preliminary results that will facilitate the subsequent asymptotic analysis. All vectors are in **boldface** to differentiate from the scalars. For a sequence of random vectors $\boldsymbol{X}^n$, we use $\boldsymbol{X}^n \to \boldsymbol{X}$ a.s., $\boldsymbol{X}^n \xrightarrow{p} \boldsymbol{X}$, and $\boldsymbol{X}^n \Rightarrow \boldsymbol{X}$ to denote almost surely point-wise convergence, convergence in probability, and convergence in distribution (weak convergence), respectively. Let $\mathcal{J} := \{1, 2, \ldots, J\}$ denote the index set of the SPs. For a vector $\boldsymbol{a} \in \mathbb{R}^J$, we use $\|\boldsymbol{a}\|$ to denote the ∞-norm, so $\|\boldsymbol{a}\| := \max_{j\in\mathcal{J}} |a_j|$. For two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^J$, we use $\langle \boldsymbol{a}, \boldsymbol{b} \rangle := \sum_{i=1}^J a_i b_i$ to represent the inner product, and use $\boldsymbol{a} \circ \boldsymbol{b} := (a_j b_j)_{j\in\mathcal{J}}$ to represent the Hadamard product. For a given nonnegative vector $\boldsymbol{\mu} \in \mathbb{R}^J_{++}$, we define the $\boldsymbol{\mu}$-norm as $\|\boldsymbol{a}\|^{\boldsymbol{\mu}} := \|\boldsymbol{a} \circ \boldsymbol{\mu}\|$. Note that the μ-norm is topologically equivalent to the ∞-norm. Let $\mathrm{Diag}\,(\boldsymbol{a})$ denote a diagonal matrix with its diagonal entries being $\boldsymbol{a}$. We use $\boldsymbol{B}(t)$ to denote a J-dimensional standard Wiener process starting at $\mathbf{0}$.

Let $D([0,+\infty),\mathbb{R}^J)$ denote the space of right-continuous functions with left limits (i.e., RCLL functions) in $\mathbb{R}^J$ with time domain $[0,+\infty)$, endowed with the usual Skorokhod topology (Jacod and Shiryaev (1987)). For any $T>0$, we define the uniform norm $\|\cdot\|_T$ on space $D([0,+\infty),\mathbb{R}^J)$ as

$$\|\boldsymbol{y}\|_T = \sup\{\|\boldsymbol{y}(t)\|,\ s\in[0,T]\}. \tag{16}$$

We denote $\|\boldsymbol{y}\|_\infty := \sup\{\|\boldsymbol{y}(s)\|,\ s\in[0,+\infty)\}$ with a slight abuse of notations. We say that $\boldsymbol{y^n}\to\boldsymbol{y}$ uniformly on all compact sets (u.o.c.), if $\|\boldsymbol{y^n}-\boldsymbol{y}\|_T\to 0$ a.s. for all $T>0$. When $\boldsymbol{y}$ is continuous, convergence in the topology induced by the uniform norm is equivalent to convergence in the Skorokhod topology Chen and Yao (2001). Therefore, to prove convergence with respect to the Skorokhod topology, it suffices to prove convergence with respect to the uniform topology on compact sets when the limit process is continuous.

We next introduce the notations of reflection mapping, which is similar to the oblique reflection mapping defined in Chapter 7 of Chen and Yao (2001) (in our model the reflection has to be normal to the surface). In this paper, we consider a rectangular domain $\Omega := \prod_{j\in\mathcal{J}}[a_j,b_j]$, with $-\infty\le\boldsymbol{a}<\boldsymbol{b}\le+\infty$. We let $(\Phi^\Omega,\Psi^\Omega,\Upsilon^\Omega)$ denote the *reflection mapping* with respect to domain Ω such that $(\Phi^\Omega,\Psi^\Omega,\Upsilon^\Omega):\ D([0,\infty),\mathbb{R}^J)\to D([0,\infty),\Omega\otimes\mathbb{R}^{2J}_+)$. We let $\boldsymbol{x}:=(x_j)_{j\in\mathcal{J}}$, $\boldsymbol{l}:=(l_j)$, and $\boldsymbol{u}:=(u_j)$ denote the image of $(\Phi^\Omega,\Psi^\Omega,\Upsilon^\Omega)$, such that

$$\begin{aligned} l_j(t) &= \sup\nolimits_{0\le s\le t}[a_j(s)+u_j(s)-z_j(s)]^+\\ u_j(t) &= \sup\nolimits_{0\le s\le t}[z_j(s)+l_j(s)-b_j(s)]^+\\ x_j(t) &= z_j(t)+l_j(t)-u_j(t). \end{aligned} \tag{17}$$

It is well know that the l_j and u_j defined as above are the minimal non-decreasing processes which enforce $x_j(t)\in[a_j,b_j]$, and satisfy the following complementary-slackness condition:

$$\begin{aligned} &\boldsymbol{x}=\boldsymbol{z}+\boldsymbol{l}-\boldsymbol{u}\\ &l_j(0)=0,\ \int_0^\infty(x_j(t)-a_j)^+dl_j(t)=0\\ &u_j(0)=0,\ \int_0^\infty(b_j-x_j(t))^+du_j(t)=0. \end{aligned} \tag{18}$$

Note that in the above definition, we allow $a_j=-\infty$ ($b_j=+\infty$), then the corresponding non-decreasing process $l_j\equiv 0$ ($u_j\equiv 0$).

To derive the fluid and diffusion limit processes, we consider a sequence of PQCDAs indexed by $n=1,2,\ldots$. In the n^{th} PQCDA, customers (including those who balk) arrive according to a time-homogeneous Poisson process with constant traffic intensity n. Upon arrival, a customer chooses the j^{th} SP with state-dependent probability $p_j(\boldsymbol{\tau}(t-))$, which is a deterministic function of the vector of waiting-time estimates $\boldsymbol{\tau}(t-)$ right before time t. We assume that the choice probabilities $\boldsymbol{\Lambda}(\boldsymbol{\tau})=(p_j(\boldsymbol{\tau}(t-)))$ satisfy all the properties given in Section 3, and are invariant with respect to the system index n. We assume that the service times for the j-th SP are i.i.d random variables with mean $1/(n\mu_j^n)$ and coefficient of variation ω_j, such that $\mu_j^n\to\mu_j$ and ω_j does not depend on n. we use $S_j^n(t)$ to represent the cumulative number of service completions at the j^{th} SP in the n^{th} PQCDA, provided that the service provider is busy in $[0,t]$. $S_j^n(t)$, as a renewal process, can be formulated as

$$S_j^n(t) := \{k \mid \sum_{i=1}^k b_j(k)\le n\mu_j^n t\}, \tag{19}$$

where $(b_j(k))$ is a sequence of i.i.d. service time random variables with mean 1, and its distribution does not depend on index n. Finally, we use $\boldsymbol{X}^n(t)$ and $\boldsymbol{\tau}^n(t)$ to denote vectors of queue-lengths and waiting-time estimates in the n^{th} PQCDA at time t, respectively, and use $W_j^n(t)$ to denote the cumulative busy time of the j^{th} SP up to time t.

6. Fluid Approximation

We first derive a compact expression for the arrival process of each queue. In the following lemma, $N(\cdot)$ denotes a rate-one standard Poisson process. $\boldsymbol{\tau}(t-)$ denotes the left limit of $\boldsymbol{\tau}(\cdot)$ at time t, which exists because $\boldsymbol{\tau}(t) = \boldsymbol{X}(t) \circ \boldsymbol{\mu}^{-1}$ is RCLL. Note that this representation does not imply that the arrival process of each queue is an independent Poisson process, because the traffic intensity is state-dependent. Similar notations have been used in existing literature (e.g., Mandelbaum et al. (1998b); Weerasinghe (2014); Dong et al. (2015)) to represent state-dependent arrival or departure processes.

Lemma 1 *The total number of customers who have joined queue j during time interval $[0,t]$ in the n^{th} PQCDA is given by $N\left(\int_0^t np_j(\boldsymbol{\tau}(s-))ds\right)$, or equivalently, $N\left(\int_0^t np_j(\boldsymbol{\tau}(s))ds\right)$.*

Although the expression provided in Lemma 1 may be intuitive, rigorous derivation of the expression relies on the Meyer's theorem (see for example, Brown and Nair (1988)) and is not straightforward. We attach the proof of Lemma 1 in Appendix B.

We next study the asymptotic behavior of the PQCDA via fluid approximation. We prove that the scaled queue-length processes in a sequence of PQCDAs converge to a *fluid limit process*. Moreover, we show that the fluid limit process converges to an equilibrium state which can be characterized as a solution to a Nonlinear-Complementarity-Problem (NCP).

In the n^{th} PQCDA, we define the scaled queue-length

$$\boldsymbol{x}^n(t) := \frac{1}{n}\boldsymbol{X}^n(t). \tag{20}$$

We next show that the process $\boldsymbol{x}^n$ converges to a fluid limit process. From hereon, without further specification, we assume that the arrival rate function $\boldsymbol{\Lambda}(\cdot) := (p_j(\cdot))$ satisfies the CD property and the stability condition (9). As a result, the Jacobean matrix of $\boldsymbol{\Lambda}(\cdot)$ is negative definite almost everywhere over $\mathbb{R}_+^J$.

Theorem 1 *(Convergence to Fluid Limit) Define $\Omega := [0, +\infty)^J$ and $\Gamma_j(\boldsymbol{x}) := p_j(\boldsymbol{x} \circ \boldsymbol{\mu}^{-1})$ for $j = 1, \ldots, J$. Suppose $\boldsymbol{x}^n(0) \to \boldsymbol{x}(0)$ a.s. when $n \to \infty$ with $\boldsymbol{x}(0) \geq 0$. Then for all $T > 0$,*

$$\|\boldsymbol{x}^n - \boldsymbol{x}\|_T \to 0, \ \ a.s. \tag{21}$$

where $\boldsymbol{x}$ is the unique solution to the following differential equation with reflection,

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}^{\Omega}\left(\boldsymbol{x}(0) + \int_0^t (\boldsymbol{\Gamma}(\boldsymbol{x}(s)) - \boldsymbol{\mu})ds\right), \tag{22}$$

where $\boldsymbol{\Phi}^{\Omega}$ is the reflecting mapping defined in Section 5.

Remark 6 *In our paper, we assume the arrival process to be time-homogeneous in order to derive steady-state characterization. The convergence to fluid limit process still holds for time-inhomogeneous arrivals and our proof can be adapted to cover this case.*

Before diving into the proof of Theorem 1, we want to comment on this result, specifically in comparison to the classical results in the literature. Mandelbaum et al. (1998b), in their Theorem 4.6, have proved that the queue-length process in a general state-dependent queueing network converges to the unique fluid limit process when the arrival and service rate functions are Lipschitz continuous. However, our customer choice model may lead to non-Lipschitz $p_j(\cdot)$ (See Remark 5). Therefore, the proof technique of Mandelbaum et al. (1998b) cannot be adapted to a proof of Theorem 1. In fact, if the drift coefficients $\boldsymbol{\Gamma}(\cdot) = (p_j(\cdot \circ \boldsymbol{\mu}^{-1}))$ in the differential equation (22) are non-Lipschitz, then generally speaking, the differential equation may not have a solution, or have multiple solutions. See the following example.

Example 6.1 *Consider a one-dimensional non-stochastic differential equation,*

$$x(t) = \int_0^t \sqrt{x(s)}\, ds. \tag{23}$$

The above equation has a non-Lipschitz drift $\sqrt{x(s)}$, and has two solutions, $x(t) \equiv 0$ and $x(t) = \frac{1}{4}t^2$. We can also provide an example of a differential equation with non-Lipschitz drift, to which a finite solution does not exist for the entire horizon.

$$x(t) = 1 + \int_0^t x^2(s)\, ds. \tag{24}$$

The above equation has a finite solution $x(t) = \frac{1}{1-t}$ only over the time window $[0,1)$.

Interestingly, we find that the choice-driven property can replace the Lipschitz condition in proving Theorem 1. To that end, the next Lemma provides a new sufficient condition for the pathwise uniqueness of a solution to the following stochastic differential equation with reflection (SDER), which is a more general form of (22) by including a stochastic term[2].

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \int_0^t \boldsymbol{b}(s, \boldsymbol{x}(s))ds + \int_0^t \boldsymbol{\sigma}(s, \boldsymbol{x}(s))d\boldsymbol{B}(s) + \boldsymbol{\ell}(t), \tag{25}$$

where $\boldsymbol{\ell}$ is a non-decreasing process that keeps $\boldsymbol{x} \geq 0$ (See Section (5) for a rigorous definition).

Lemma 2 *Suppose $\boldsymbol{b}(s,\cdot)$ is absolute continuous with negative definite Jacobean matrix a.e., and $\boldsymbol{\sigma}(s,\cdot)$ is Lipschitz continuous for all s, that is, $\|\boldsymbol{\sigma}(s,\boldsymbol{x}) - \boldsymbol{\sigma}(s,\boldsymbol{y})\| \leq K\|\boldsymbol{x}-\boldsymbol{y}\|$ for some constant $K > 0$. Then the solution to SDER* (25)*, if exists, must be pathwise unique.*

Proof. Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are both solutions to SDER (25). Then by the first equation in the proof of Theorem 4.1 in (Tanaka (1979), page 175), we have

$$\begin{aligned} &\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|^2 \\ &\leq \|\textstyle\int_0^t (\boldsymbol{\sigma}(s,\boldsymbol{x}(s)) - \boldsymbol{\sigma}(s,\boldsymbol{y}(s)))d\boldsymbol{B}(s)\|^2 + 2\int_0^t \langle \boldsymbol{x}(s) - \boldsymbol{y}(s), \boldsymbol{b}(s,\boldsymbol{x}(s)) - \boldsymbol{b}(s,\boldsymbol{y}(s))\rangle ds + \text{ the remainder.} \end{aligned} \tag{26}$$

where the remainder has zero expectation. We thus have

$$\begin{aligned} &\mathbb{E}\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|^2 \\ &\leq \mathbb{E}\textstyle\int_0^t \|\boldsymbol{\sigma}(s,\boldsymbol{x}(s)) - \boldsymbol{\sigma}(s,\boldsymbol{y}(s))\|^2 ds + 2\int_0^t \langle \boldsymbol{x}(s) - \boldsymbol{y}(s),\ \boldsymbol{b}(s,\boldsymbol{x}(s)) - \boldsymbol{b}(s,\boldsymbol{y}(s))\rangle\, ds \\ &\leq K^2\mathbb{E}\textstyle\int_0^t \|\boldsymbol{x}(s) - \boldsymbol{y}(s)\|^2 ds + 2\int_0^t \langle \boldsymbol{x}(s) - \boldsymbol{y}(s),\ \boldsymbol{b}(s,\boldsymbol{x}(s)) - \boldsymbol{b}(s,\boldsymbol{y}(s))\rangle\, ds \end{aligned} \tag{27}$$

where the inequality follows from Lipschitz continuity of $\boldsymbol{\sigma}(s,\cdot)$. By absolute continuity of $\boldsymbol{b}(s,\cdot)$, we have

$$\boldsymbol{b}(s,\boldsymbol{x}(s)) - \boldsymbol{b}(s,\boldsymbol{y}(s)) = \int_0^1 \boldsymbol{R}(\boldsymbol{y}(s) + \xi(\boldsymbol{x}(s) - \boldsymbol{y}(s)))(\boldsymbol{x}(s) - \boldsymbol{y}(s))\, d\xi, \tag{28}$$

with the Jacobean matrix $\boldsymbol{R}(\boldsymbol{y}(s) + \xi(\boldsymbol{x}(s) - \boldsymbol{y}(s)))$ negative definite for almost all $\xi \in [0,1]$. Consequently,

$$\begin{aligned} \langle \boldsymbol{x}(s) - \boldsymbol{y}(s), \boldsymbol{b}(s,\boldsymbol{x}(s)) - \boldsymbol{b}(s,\boldsymbol{y}(s))\rangle &= \langle \boldsymbol{x}(s) - \boldsymbol{y}(s),\ \textstyle\int_0^1 \boldsymbol{R}(\boldsymbol{y}(s) + \xi(\boldsymbol{x}(s) - \boldsymbol{y}(s)))(\boldsymbol{x}(s) - \boldsymbol{y}(s))\, d\xi\rangle \\ &= \textstyle\int_0^1 \langle \boldsymbol{x}(s) - \boldsymbol{y}(s),\ \boldsymbol{R}(\boldsymbol{y}(s) + \xi(\boldsymbol{x}(s) - \boldsymbol{y}(s)))(\boldsymbol{x}(s) - \boldsymbol{y}(s))\rangle\, d\xi \\ &\leq 0 \end{aligned} \tag{29}$$

[2] For the purpose of proving Theorem 1, we only need a weaker version of Lemma 2 that deals with a non-stochastic differential equation with reflection. We presented Lemma 2 as a general result on SDER, because of its independent interest.

which, together with Equation (27), leads to

$$\mathbb{E}\|\boldsymbol{x}(t)-\boldsymbol{y}(t)\|^2 \leq K^2 \int_0^t \mathbb{E}\|\boldsymbol{x}(s)-\boldsymbol{y}(s)\|^2 ds. \tag{30}$$

Then by the Gronwall's inequality (e.g., Ethier and Kurtz (2009), page 498), we have $\|\boldsymbol{x}(t)-\boldsymbol{y}(t)\| = 0$. ■

Tanaka (1979) and Dupuis and Ishii (1993) proved that there exists a pathwise unique solution to (25) if both $\boldsymbol{b}(s,\cdot)$ and $\boldsymbol{\sigma}(s,\cdot)$ are Lipschitz continuous. Swart (2002) and Yamada and Watanabe (1971) discussed pathwise uniqueness under some similar but more general conditions. While our Lemma 2 states that the Lipschitz continuity of the drift coefficient $\boldsymbol{b}(s,\cdot)$ can be replaced by absolute continuity with negative definite Jacobean a.e. Our result thus complements the existing results on pathwise uniqueness of the solution to (25).

As a notable difference from the standard proof, the proof for Theorem 1 also leverages the CD property instead of the Lipschitz property of the arrival rate function. To leverage the CD property, the proof invokes the inequalities of SDERs in Tanaka (1979) rather than directly applying the Gronwall's inequality.

Proof of Theorem 1 By Lemma 1, the length of queue j is described by the following equation,

$$\begin{aligned} x_j^n(t) &= x_j^n(0) + \tfrac{1}{n}N(\textstyle\int_0^t np_j(\boldsymbol{\tau}^n(s))ds) - \tfrac{1}{n}S_j^n(W_j^n(t)) \\ &= x_j^n(0) + \tfrac{1}{n}Z_j^n(t) + \textstyle\int_0^t (p_j(\boldsymbol{x}^n(s)\circ(\boldsymbol{\mu}^n)^{-1}) - p_j(\boldsymbol{x}^n(s)\circ\boldsymbol{\mu}^{-1}))ds \\ &\quad + \textstyle\int_0^t (p_j(\boldsymbol{x}^n(s)\circ\boldsymbol{\mu}^{-1}) - \mu_j^n)ds + \ell_j^n(t) \end{aligned} \tag{31}$$

where $x_j^n(t)$ was defined in (20), $\ell_j^n(t) := \mu_j^n(t - W_j^n(t))$ is the minimal non-decreasing process which ensures $x_j^n(t) \geq 0$, and

$$\begin{aligned} Z_j^n(t) := &\Big(N(\textstyle\int_0^t np_j(\boldsymbol{X}^n(s)\circ(n\boldsymbol{\mu}^n)^{-1})ds) - \int_0^t np_j(\boldsymbol{X}^n(s)\circ(n\boldsymbol{\mu}^n)^{-1})ds\Big) \\ &+ \big(n\mu_j^n W_j^n(t) - S_j^n(W_j^n(t))\big) \end{aligned} \tag{32}$$

represents a mean-zero centered process. We also define $\boldsymbol{\Gamma}(\boldsymbol{x}) := \boldsymbol{\Lambda}(\boldsymbol{x}\circ\boldsymbol{\mu}^{-1})$ and

$$\tilde{\boldsymbol{z}}^n(t) := \tfrac{1}{n}\boldsymbol{Z}^n(t) + \textstyle\int_0^t (\boldsymbol{\Lambda}(\boldsymbol{x}^n(s)\circ(\boldsymbol{\mu}^n)^{-1}) - \boldsymbol{\Lambda}(\boldsymbol{x}^n(s)\circ(\boldsymbol{\mu})^{-1}))ds. \tag{33}$$

Then we can express $\boldsymbol{x}(t)$ and $\boldsymbol{x}^n(t)$ as

$$\begin{aligned} \boldsymbol{x}^n(t) &= \boldsymbol{x}^n(0) + \textstyle\int_0^t \boldsymbol{\Gamma}(\boldsymbol{x}^n(s))ds - t\boldsymbol{\mu}^n + \tilde{\boldsymbol{z}}^n(t) + \boldsymbol{\ell}^n(t), \\ \boldsymbol{x}(t) &= \boldsymbol{x}(0) + \textstyle\int_0^t \boldsymbol{\Gamma}(\boldsymbol{x}(s))ds - t\boldsymbol{\mu} + \boldsymbol{\ell}(t). \end{aligned} \tag{34}$$

where $\boldsymbol{\ell}(\cdot) := (\ell_j(\cdot))_{j=1,\ldots,J}$ and $\boldsymbol{\ell}^n(\cdot)$ denote the minimal non-decreasing processes that keep $\boldsymbol{x}(t)$ and $\boldsymbol{x}^n(t)$ staying non-negative.

We invoke the first inequality in Remark 2.2 of (Tanaka, 1979), in which we plug in the following quantity $\xi(t) := \boldsymbol{x}(t)$, $\tilde{\xi}(t) := \boldsymbol{x}^n(t)$, $w(t) := \boldsymbol{x}(0) - t\boldsymbol{\mu}$ and $\tilde{w}(t) := \boldsymbol{x}^n(0) + \tilde{\boldsymbol{z}}^n(t) - t\boldsymbol{\mu}^n$, $a(t) = \boldsymbol{\Gamma}(\boldsymbol{x}(t))$ and $\tilde{a}(t) = \boldsymbol{\Gamma}(\boldsymbol{x}^n(t))$. Since $\boldsymbol{\Gamma}(\cdot)$ is absolutely continuous, $a(\cdot)$ and $\tilde{a}(\cdot)$ are both right continuous and have bounded variation, which satisfy the conditions specified in (Tanaka, 1979). The first inequality in Remark 2.2 of (Tanaka, 1979) then leads to following inequality,

$$\begin{aligned} &\|\boldsymbol{x}^n(t) - \boldsymbol{x}(t)\|^2 \\ \leq\ &\|\boldsymbol{x}^n(0) - \boldsymbol{x}(0) + \tilde{\boldsymbol{z}}^n(t) - t(\boldsymbol{\mu}^n - \boldsymbol{\mu})\|^2 + 2\textstyle\int_0^t \langle \boldsymbol{x}^n(s) - \boldsymbol{x}(s), \boldsymbol{\Gamma}(\boldsymbol{x}^n(s)) - \boldsymbol{\Gamma}(\boldsymbol{x}(s))\rangle ds \\ &+ \textstyle\int_0^t \langle \tilde{\boldsymbol{z}}^n(t) - \tilde{\boldsymbol{z}}^n(s) - (\boldsymbol{\mu}^n - \boldsymbol{\mu})(t-s), d\tilde{a}(s) - da(s) + d\tilde{\boldsymbol{\ell}}(s) - d\boldsymbol{\ell}(s)\rangle ds \end{aligned} \tag{35}$$

Later, we will prove that $\|\tilde{\boldsymbol{z}}^n\|_T \to 0$ for all $T > 0$. Since $\|\boldsymbol{x}^n(0) - \boldsymbol{x}(0)\| \to 0$ and $\|\boldsymbol{\mu}^n - \boldsymbol{\mu}\| \to 0$, the first and the third terms on the right-hand-side of Equation (35) both converge to zero. The second term is non-positive because

$$\begin{aligned}
&\langle \boldsymbol{x}^n(s) - \boldsymbol{x}(s), \boldsymbol{\Gamma}(\boldsymbol{x}^n(s)) - \boldsymbol{\Gamma}(\boldsymbol{x}(s))\rangle \\
&= \langle \boldsymbol{x}^n(s) - \boldsymbol{x}(s), \textstyle\int_0^1 \boldsymbol{R}(\boldsymbol{x}^n(s) + \xi(\boldsymbol{x}^n(s) - \boldsymbol{x}(s)))((\boldsymbol{x}^n(s) - \boldsymbol{x}(s)) \circ \boldsymbol{\mu}^{-1})\rangle \\
&= \langle (\boldsymbol{x}^n(s) - \boldsymbol{x}(s)) \circ \boldsymbol{\mu}^{-1/2}, \textstyle\int_0^1 \boldsymbol{R}(\boldsymbol{x}^n(s) + \xi(\boldsymbol{x}^n(s) - \boldsymbol{x}(s)))((\boldsymbol{x}^n(s) - \boldsymbol{x}(s)) \circ \boldsymbol{\mu}^{-1/2})\rangle \\
&\le 0,
\end{aligned} \tag{36}$$

where the last inequality follows from that the Jacobean matrix $\boldsymbol{R}(\boldsymbol{x}^n(s) + \xi(\boldsymbol{x}^n(s) - \boldsymbol{x}(s)))$ is negative semidefinite a.e. The inequality (35) thus implies that $\|\boldsymbol{x}^n(t) - \boldsymbol{x}(t)\|^2 \to 0$.

It remains to show that $\|\tilde{\boldsymbol{z}}^n\|_T \to 0$ for all fixed $T > 0$. By the functional strong law of large number (e.g., Theorem 5.10 in Chen and Yao (2001)), and $\boldsymbol{\mu^n} \to \boldsymbol{\mu}$, we have

$$\begin{aligned}
\tfrac{1}{n}\|N(n \textstyle\int_0^t p_j(\frac{\boldsymbol{X}_j^n(s)}{n\mu_j^n})ds) - \int_0^t np_j(\frac{\boldsymbol{X}_j^n(s)}{n\mu_j^n})ds\|_T &\to 0 \\
\tfrac{1}{n}\|n\mu_j^n W_j^n(t) - S_j^n(W_j^n(t))\|_T &\to 0.
\end{aligned} \tag{37}$$

We thus conclude that

$$\|\frac{1}{n}\boldsymbol{Z}^n\|_T \to 0. \tag{38}$$

Also, since $\boldsymbol{\Lambda}(\cdot)$ is continuous and bounded (by one), by bounded convergence, we have

$$\|\int_0^t (\boldsymbol{\Lambda}(\boldsymbol{x}^n(s) \circ (\boldsymbol{\mu}^n)^{-1}) - \boldsymbol{\Lambda}(\boldsymbol{x}^n(s) \circ (\boldsymbol{\mu})^{-1}))ds\|_T \le \int_0^T \|\boldsymbol{\Lambda}(\boldsymbol{x}^n(s) \circ (\boldsymbol{\mu}^n)^{-1}) - \boldsymbol{\Lambda}(\boldsymbol{x}^n(s) \circ (\boldsymbol{\mu})^{-1}))\|ds \to 0 \tag{39}$$

Equations (38) and (39) imply that $\|\tilde{\boldsymbol{z}}^n\|_T \to 0$.

■

We call $\boldsymbol{x}$ the *fluid limit process* of the PQCDA. Because there is a one-to-one correspondence between $\boldsymbol{X}(t)$ and $\boldsymbol{\tau}(t)$ via equation (8), we can alternatively represent the fluid limit process using $\{\boldsymbol{\tau}(t) : t \ge 0\}$. We next define the equilibrium (stationary) state of this fluid limit process.

Definition 2 $\boldsymbol{x}^* := (\boldsymbol{x}_j^*) \in \mathbb{R}_+^J$ *is an equilibrium queue-length vector if given* $\boldsymbol{x}(0) = \boldsymbol{x}^*$*, the differential equation* (22) *has the solution* $\boldsymbol{x}(t) \equiv \boldsymbol{x}^*$*. The associated* $\boldsymbol{\tau^*} := (\tau_j^*) = (x_j^*/\mu_j)$ *is referred to as an equilibrium waiting-time vector.*

Intuitively, a fluid limit process is at an equilibrium state if and only if the net flow rate (i.e., difference between the arrival and departure rates) equals to zero for each queue. This logic leads to the following characterization of an equilibrium state.

Proposition 2 $\boldsymbol{\tau}^*$ *is an equilibrium waiting-time vector of an PQCDA if and only if* $\boldsymbol{\tau}^*$ *is the solution to the following nonlinear complementarity problem (NCP):*

$$NCP \qquad \begin{aligned}
\mu_j - p_j(\boldsymbol{\tau}) &\ge 0, \quad \textit{for } j = 1, \ldots, J. \\
\tau_j &\ge 0 \quad \textit{for } j = 1, \ldots, J. \\
\textstyle\sum_{j=1}^J \tau_j(\mu_j - p_j(\boldsymbol{\tau})) &= 0.
\end{aligned} \tag{40}$$

The proof of Proposition 2 is attached in Appendix C.

Theorem 2 *(Existence and Uniqueness of Equilibrium) There exists a unique equilibrium waiting-time vector* $\boldsymbol{\tau}^*$ *for the fluid limit process in each PQCDA.*

It suffices to prove that the NCP (40) always has a unique solution. To that end, we prove that $-\mathbf{\Lambda}(\cdot)$ satisfies the so-called P-property (Moré and Rheinboldt, 1973), which implies uniqueness. We then construct a solution to the NCP via a tatonnement process, i.e., by adjusting the value of τ_j according to the demand-supply gap $\mu_j - p_j(\boldsymbol{\tau})$. A complete proof is provided in Appendix D.

Remark 7 *Since our proof for the existence of an NCP solution is constructive, the tatonnement algorithm introduced in the proof can be used to calculate the equilibrium queue-length vector (or equilibrium waiting-time vector).*

The choice-driven property is not only sufficient for the existence and uniqueness of the equilibrium state, but also necessary in the sense that without it, these results cannot hold for *certain* parameters. Please see the following examples as an illustration of this point.

Example 6.2 *This example shows that when (CD-a) is violated, the fluid limit process may have multiple equilibria. Consider an example with* $\boldsymbol{\mu} = (0.4, 0.4)^T$*, arrival rate function* $\mathbf{\Lambda}(\boldsymbol{\tau}) = (0.4 - 0.1\exp(-(\tau_1 - 1)^2), 0.4 - 0.1\exp(-(\tau_2 - 1)^2)^T$*, and its Jacobean* $\boldsymbol{R}(\boldsymbol{\tau}) = \begin{pmatrix} 0.2(\tau_1 - 1)\exp(-(\tau_1 - 1)^2) & 0 \\ 0 & 0.2(\tau_2 - 1)\exp(-(\tau_2 - 1)^2) \end{pmatrix}$*. The* j^{th} *diagonal elements are positive when* $\tau_j < 1$*, so (CD-a) is violated. For queue* $j = 1, 2$*, the maximum arrival rate is attained when* $\tau_j = 1$*, at which time the arrival rate and service rate is balanced. Thus,* $\tau_j = 1$ *is an equilibrium queue length for each queue. In addition to that,* $\tau_j = 0$ *is also an equilibrium queue length. Thus, this PQCDA consists four equilibrium states,* $(1,1)^T$*,* $(0,1)^T$*,* $(1,0)^T$*,* $(0,0)^T$*.*

Example 6.3 *This example shows that an equilibrium state may not exist when (CD-b) is violated. Consider an example with* $\boldsymbol{\mu} = (1, 0.01, 0.01)^T$*,* $\boldsymbol{R} \equiv \begin{pmatrix} -0.2 & -0.1 & -0.1 \\ -0.1 & -0.1 & 0.15 \\ -0.1 & 0.15 & -0.1 \end{pmatrix}$*, and* $\mathbf{\Lambda}(\boldsymbol{\tau}) = (\boldsymbol{R}\boldsymbol{\tau})^+$*. This example satisfies the stability condition* (9)*, because the arrival rate for each queue converges to zero when its length approaches to infinity, as long as the lengths of the other queues are fixed. The Jacobean also contains negative diagonals and has negative row and column sums, so (CD-a) and (CD-c) are both satisfied. However, the Jacobean matrix contains negative off-diagonal entries and therefore violates assumption (CD-b). One can check that if the fluid limit process starts from* $(0,1,1)^T$*, then we will have* $\tau_1(t) \equiv 0$*, and* $\tau_2(t) \equiv \tau_3(t) \to \infty$ *when* $t \to \infty$*. Consequently, no equilibrium exists.*

Example 6.4 *This example shows that when (CD-c) is violated, the fluid limit process may also have multiple equilibria. Consider an PQCDA has* $\boldsymbol{R} \equiv \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$*,* $\boldsymbol{\mu} = (0.5, 0.5)^T$*,* $\mathbf{\Lambda}(\boldsymbol{\tau}) = ((0.5, 0.5)^T + \boldsymbol{R}\boldsymbol{\tau})^+$*. Then any vector in the form of* $(z, z)^T$ *with* $z \geq 0$ *can be an equilibrium state of the fluid limit process.*

The above examples show that when any one of (CD-a), (CD-b), (CD-c) fails, the fluid limit process may not have a unique equilibrium state. Because all the subsequent asymptotic characterizations for the PQCDA (e.g., convergence of the fluid limit process to the equilibrium, convergence to the diffusion limit process, and the stationary distribution of the diffusion limit process) rely on the fact that the fluid limit process has a unique equilibrium state, these characterizations would not apply to general parallel-queue systems.

The next theorem shows that given the CD property, the fluid limit of the expected delays in PQCDA must converge to the unique equilibrium state.

Theorem 3 *(Convergence to Equilibrium) Suppose* $\{\boldsymbol{x}(t) | t \geq 0\}$ *is a solution to the differential equation* (22) *with* $\boldsymbol{x}(0) \geq 0$*, and* $\boldsymbol{\tau}(t) = \boldsymbol{x}(t) \circ \boldsymbol{\mu}^{-1}$*. Then*

$$\boldsymbol{\tau}(t) \to \boldsymbol{\tau}^*, \qquad \text{when } t \to \infty. \tag{41}$$

The main idea of the proof involves showing the maximal deviation from the equilibrium queue length $\max_j \tau_j(t) - \tau_j^*$ decreases with time due to the CD properties. A complete proof is provided in Appendix E.

7. Diffusion Approximation

In contrast to the fluid model, a diffusion process can capture the asymptotic behavior of the queue-length process at a more granular level. We show that when the queue lengths are close to the equilibrium, then its deviation from the equilibrium, under diffusion scaling, converges to a diffusion limit which is known as a reflected multi-dimensional Ornstein-Uhlenbeck (RMOU) process. We continue to examine the sequence of PQCDAs defined in Section 3. In the n^{th} PQCDA, we define the *virtual equilibrium* $\boldsymbol{\tau}^{n,*}$ as the solution to the following NCP

$$\text{NCP} \qquad \begin{array}{rl} n\mu_j^n - np_j(\boldsymbol{\tau}^{n,*}) \geq 0, & \text{for } j = 1, \ldots, J. \\ \tau_j^{n,*} \geq 0 & \text{for } j = 1, \ldots, J. \\ \sum_{j=1}^J \tau_j^{n,*}(n\mu_j^n - np_j(\boldsymbol{\tau}^{n,*})) = 0. & \end{array} \tag{42}$$

The *virtual equilibrium* can be interpreted as a state at which the mean arrival rate and service rate are balanced in each queue in the n^{th} PQCDA. Since we have assumed that $\mu_j^n \to \mu_j$, the continuity of $p_j(\boldsymbol{\tau})$ implies that the limit of $\boldsymbol{\tau}^{n,*}$ must solve the NCP (40) for the fluid model. Since the solution to (40) is unique according to Theorem 2, we deduce that $\boldsymbol{\tau}^{n,*} \to \boldsymbol{\tau}^*$. We use $\rho_j^n := \frac{p_j(\boldsymbol{\tau}^{n,*})}{\mu_j^n}$ to denote the traffic intensity at the equilibrium waiting-times. Correspondingly, we denote the traffic intensity of queue j in the fluid model by $\rho_j := \lim_{n\to\infty} \rho_j^n$. We consider four mutually exclusive cases of the limiting behaviors of the sequences (τ_j^n) and (ρ_j^n). Note that ρ_j^n is no greater than one in all queues by the NCP condition. $\tau_j^{n,*} > 0$ implies that $\rho_j^n = 1$ by complementarity slackness. These four cases are not exhaustive, but they cover the scenarios which have been most often considered in the literature (e.g., Ward and Glynn (2003)).

$$\begin{array}{ll} \textit{Largely Under-demand Queues} & \mathcal{J}^{--} := \{j | \rho_j^n \to \rho_j < 1\} \\[2ex] \textit{Balanced or Slightly Under-demand Queues} & \mathcal{J}^{-} := \{j | \begin{array}{l} \tau_j^{n,*} = 0,\ \rho_j^n \leq 1 \text{ for all } n,\ \rho_j^n \to 1, \\ \sqrt{n}(\mu_j^n - p_j(\boldsymbol{\tau}^{n,*})) \to \theta_j \geq 0 \end{array} \} \\[2ex] \textit{Slightly Over-demand Queues} & \mathcal{J}^{+} := \{j | \begin{array}{l} \tau_j^{n,*} > 0 \text{ for all } n,\ \tau_j^{n,*} \to \tau_j^* = 0, \\ \sqrt{n}(\mu_j^n \tau_j^{n,*} - \mu_j \tau_j^*) \to \vartheta_j \geq 0 \end{array} \} \\[2ex] \textit{Largely Over-demand Queues} & \mathcal{J}^{++} := \{j | \begin{array}{l} \tau_j^{n,*} \to \tau_j^* > 0, \\ \sqrt{n}(\mu_j^n \tau_j^{n,*} - \mu_j \tau_j^*) \to \vartheta_j \end{array} \}, \end{array} \tag{43}$$

where $\boldsymbol{\vartheta} := (\vartheta_j)$ and $\boldsymbol{\theta} := (\theta_j)$ are both J-dimensional vectors and have the following expressions,

$$\theta_j = \begin{cases} \lim_{n\to\infty} \sqrt{n}(\mu_j^n - p_j(\boldsymbol{\tau}^{n,*})) & \text{if } j \in \mathcal{J}^- \\ 0 & \text{otherwise,} \end{cases} \quad \vartheta_j = \begin{cases} \lim_{n\to\infty} \sqrt{n}(\mu_j^n \tau_j^{n,*} - \mu_j \tau_j^*) & \text{if } j \in \mathcal{J}^+ \cup \mathcal{J}^{++} \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

We next investigate the diffusion approximation for the scaled queue-length process

$$Q_j^n(t) := \sqrt{n}(x_j^n(t) - x_j^*), \tag{45}$$

where $\boldsymbol{x}^n$ represents the queue-lengths under fluid scaling that has been defined in Equation (20), and $x_j^* = \mu_j \tau_j^*$ gives the length of queue j at the virtual equilibrium. For largely under-demand queues where $\rho_j < 1$, it is known that there is no diffusion for those queues, i.e., $\boldsymbol{Q}_j^n \Rightarrow 0$ (see e.g. Choudhury et al. (1997)). Therefore we can assume that $\mathcal{J}^{--} = \emptyset$ without loss of generality, as those queues have constant length of zero under diffusion scaling. We can focus on characterizing the asymptotic behavior of the scaled queue-length process for queues in $\mathcal{J}^-$, $\mathcal{J}^+$, and $\mathcal{J}^{++}$, which can co-exist in the same system. For $j \in \mathcal{J}^- \cup \mathcal{J}^+$, we have $x_j^* = \mu_j \tau_j^* = 0$ and thus $Q_j^n(t) \geq 0$; for $j \in \mathcal{J}^{++}$, since $x_j^* > 0$, $Q_j^n(t)$ can be either positive or negative. Consequently, $\boldsymbol{Q}^n$ and its diffusion limit process $\boldsymbol{Y}$ must reside in the following domain:

$$\Omega = \otimes[0, +\infty)^{J^- + J^+} \otimes (-\infty, +\infty)^{J^{++}}. \tag{46}$$

For the diffusion limit process to exist, we need to assume that the arrival rate function to have a finite Jacobean matrix $\boldsymbol{R}^*$ at the equilibrium $\boldsymbol{\tau}^*$. This assumption, however, is without loss of generality as the choice-driven property states that a finite Jacobean exists a.e. Under this assumption, we derive the diffusion limit for the queue-lengths process in PQCDA as the solution to the following SDER,

$$\boldsymbol{Y}(t) = \int_0^t \big(\boldsymbol{R}^* \mathrm{Diag}\,(\boldsymbol{\mu}^{-1})(\boldsymbol{Y}(s) - \boldsymbol{\vartheta}) - \boldsymbol{\theta}\big)\, ds + \boldsymbol{\Sigma} \boldsymbol{B}(t) + \boldsymbol{L}(t), \tag{47}$$

where $\boldsymbol{\Sigma}$ is a J-by-J diagonal matrix with $\sqrt{(1+\omega_j^2)\mu_j}$ as its j^{th} diagonal entry, $\boldsymbol{B}(t)$ is a J-dimensional standard Brownian motion with covariance matrix I (identify matrix), and $\boldsymbol{L}(t)$ is a J-dimensional minimal non-decreasing process which makes $Y_j(t) \geq 0$ for all $j \in \mathcal{J}^- \cup \mathcal{J}^+$.

Theorem 4 *(Convergence to Diffusion Limit) Suppose* $\boldsymbol{Q}^n(0) \Rightarrow \boldsymbol{Y}(0)$ *and* $\mathbb{E}\|\boldsymbol{Y}(0)\| < \infty$. *We then have,*

$$\boldsymbol{Q}^n \Rightarrow \boldsymbol{Y}. \tag{48}$$

Before proving the above result, we make a few remarks. First, according to Theorem 4, the diffusion process has a reflection barrier at 0 only for $j \in \mathcal{J}^- \cup \mathcal{J}^+$, but has no reflection barrier for $j \in \mathcal{J}^{++}$. Intuitively, for $j \in \mathcal{J}^- \cup \mathcal{J}^+$, we have $Q_j^n(t) = \sqrt{n} x_j^n(t)$. Thus, $Q_j^n(t) = 0$ (so $x_j^n(t) = 0$) means that queue j is empty, at which time the server has to stop working and prevents $Q_j^n(t)$ from decreasing further. Therefore, if $j \in \mathcal{J}^- \cup \mathcal{J}^+$, 0 is a reflecting barrier for $Q_j^n(t)$. For $j \in \mathcal{J}^{++}$, since $x_j^* = \mu_j \tau_j^* > 0$, an empty queue $(x_j^n(t) = 0)$ corresponds to $Q_j^n(t) = \sqrt{n}(0 - x_j^*) \to -\infty$ when $n \to \infty$. That means, if $j \in \mathcal{J}^{++}$, the reflection barrier for $Q_j^n(t)$ is at $-\infty$, which is equivalent to the case of no reflection barrier.

Second, we provide some interpretations of the two vectors $\boldsymbol{\theta}$ and $\boldsymbol{\vartheta}$ in Equation (44). For $j \in \mathcal{J}^-$, $\vartheta_j = 0$, while $-\theta_j$ represents the negative drift that brings down $Q_j^n(t)$ towards zero, due to the fact that the center of the RMOU is actually negative along the j^{th} coordinate. For $j \in \mathcal{J}^+ \cup \mathcal{J}^{++}$, $\theta_j = 0$, and ϑ_j can be considered as the center of the RMOU for queues along the j^{th} coordinate. Figure 2 depicts the behavior of Y_j and illustrates the role of $\boldsymbol{\theta}$ and $\boldsymbol{\vartheta}$ in the cases when j is in $\mathcal{J}^-$, $\mathcal{J}^+$, and $\mathcal{J}^{++}$, respectively.

Finally, we want to elaborate on the relationship between our result and Theorem 7.2 in Mandelbaum et al. (1998b). Mandelbaum et al. (1998b) developed a diffusion approximation for $\sqrt{n}(\boldsymbol{x}^n(t) - \boldsymbol{x}(t))$, which is the deviation of the scaled queue lengths from the fluid limit amplified by $\sqrt{n}$. The same result, nevertheless, cannot be expected in our model. This is because the drift coefficients $\boldsymbol{R}(\boldsymbol{\tau}(t))$ in the SDER (47) may have infinite values when the fluid limit $\boldsymbol{x}(t)$ passes through points at which a finite-valued Jacobean matrix does not exist. Should that happen, the sequence of $\boldsymbol{Q}^n$ may be not tight and the diffusion limit is not well defined. Thus, for our model, the diffusion limit can only be developed in a neighborhood of the fixed equilibrium state $\boldsymbol{x}^*$, at which a finite Jacobean is assumed to exist.

To develop a diffusion approximation for $\sqrt{n}(\boldsymbol{x}^n(t) - \boldsymbol{x}^*)$, we assume that the fluid limit starts with the steady state (i.e., $\boldsymbol{x}(0) = \boldsymbol{x}^*$, or more strongly, $\boldsymbol{Q}^n(0)$ converges to a bounded random variable). Then by the definition of equilibrium, we know the fluid limit is invariant as $\boldsymbol{x}(t) \equiv \boldsymbol{x}^*$. Therefore, we actually developed a diffusion approximation for the deviation of the scaled queue length from its fluid limit. Moreover, in our model, the drift coefficient in the diffusion limit is the net flow rate at the equilibrium, which allows an affine approximation using the Jacobean at the equilibrium $\boldsymbol{R}^*$. So we can derive the diffusion limit as an RMOU process, which has a stationary distribution due to negative definiteness of $\boldsymbol{R}^*$. Such a result, however, cannot be expected in a general state-dependent queueing network, because the fluid limit there may not has an equilibrium, and the drift function would not exhibit similar properties (i.e., can be approximated by an affine function with negative definite coefficient matrix).

We also wish to emphasize that the framework introduced in Theorem 7.2 of Mandelbaum et al. (1998b) cannot be adapted to derive our Theorem 4, even by assuming $\boldsymbol{x}(0) = \boldsymbol{x}^*$ in their proof. This is because their proof framework heavily relies on the bounded derivative (or Lipschitz continuity) condition for the state-dependent net flow rates. Without the Lipschitz condition, several of their intermediate results cannot hold in general, including their Lemma 14.12 (compact containment), Lemma 14.13 (C-tightness), and Lemma 14.14 (characterization of the limit process); while those results are all needed for their proof of Theorem 7.2. In particular, their Lemma 14.12 states that for all $T > 0$, $\{\boldsymbol{Q}^n(t) | t \in [0, T]\}$, as defined in (45), will be contained in a compact set with probability approaching to one when $n \to \infty$. This conclusion, nevertheless, is not valid if the arrival rates (thus the drift coefficients) are non-Lipschitz. To see this, recall the example we gave in the differential equation (24), in which the drift coefficient is non-Lipschitz and its solution becomes infinitely large for $t \in [1, +\infty)$. Although that differential equation is non-stochastic, adding a stochastic term will not change the boundedness of the solution. Therefore, non-Lipschitz arrival rates, if without additional constraints, may lead to a queue-lengths process that violates the compact containment condition.

To deal with the non-Lipschitz case, it suffices to prove a result analogous to Lemma 14.12 (compact containment) of Mandelbaum et al. (1998b) in Lemma 3, but for non-Lipschitz and choice-driven arrival rates. With compact containment, we can find a compact neighborhood of the equilibrium which contains the scaled stochastic processes at almost all the times for sufficiently large n. Since the drift function is Lipschitz continuous in that neighborhood, the convergence to the diffusion limit follows from Theorem 7.2 in Mandelbaum et al. (1998b).

Below we provide more details about the compact containment result. For a given $\kappa > 0$, we define a compact rectangular

$$\Omega(\kappa) := [0, +\kappa]^{J^- \cup J^+} \otimes [-\kappa, +\kappa]^{J^{++}}. \tag{49}$$

Define a bounded modification of $\boldsymbol{Q}^n$ as

$$\boldsymbol{Q}^{\kappa,n}(t) = \boldsymbol{\Phi}^{\Omega(\kappa)}(\boldsymbol{Q}^n) \tag{50}$$

Intuitively, $\boldsymbol{Q}^{\kappa,n}$ is the process created from $\boldsymbol{Q}^n$ by imposing reflection barriers on the finite boundary of $\Omega(\kappa)$. We prove that in the following lemma that for any $T > 0$, when $\kappa \to \infty$, with probability approaching one, $\boldsymbol{Q}^{\kappa,n}$ is contained in the bounded rectangular $\Omega(\kappa)$.

Lemma 3 *(Compact Containment) For any $T > 0$, $\epsilon > 0$, when $\kappa \to \infty$, we have*

$$\limsup_{n\to\infty} \Pr(\|\boldsymbol{Q}^n\|_T > \kappa) = \limsup_{n\to\infty} \Pr(\|\boldsymbol{Q}^{\kappa,n} - \boldsymbol{Q}^n\|_T \neq 0) \to 0 \tag{51}$$

To provide some intuition behind the proof of Lemma 3, we note that without the Lipschitz assumption, a small deviation of $\boldsymbol{Q}^n$ might lead to a large drift that pushes $\boldsymbol{Q}^n$ away from the equilibrium, which causes compact containment to fail. However, the choice-driven property ensures that any deviation of $\boldsymbol{Q}^n$ can only result in a drift that pulls $\boldsymbol{Q}^n$ back towards the equilibrium (even though the drift can be quite large). Thus, the choice-driven property can replace the Lipschitz condition and guarantee compact containment of $\boldsymbol{Q}^n$. A complete proof is provided in Appendix F. With Lemma 3, we prove Theorem 4 as follows.

Proof of Theorem 4 Since $\|\boldsymbol{Q}^{\kappa,n}\|_T \leq \kappa$, if we define the waiting-time vector associated with $\boldsymbol{Q}^{\kappa,n}$ as

$$\boldsymbol{\tau}^{\kappa,n}(t) = (n^{1/2}\boldsymbol{Q}^{\kappa,n}(t) + n\boldsymbol{\tau}^* \circ \boldsymbol{\mu}^*) \circ (n\boldsymbol{\mu}^n)^{-1}, \tag{52}$$

then $\|\boldsymbol{\tau}^{\kappa,n} - \boldsymbol{\tau}^*\|_T \to 0$. We can then select a neighborhood $\mathcal{N}$ of $\boldsymbol{\tau}^*$, such that $\boldsymbol{\tau}^{\kappa,n} \in \mathcal{N}$ for all sufficiently large n, and the arrival rate function $\boldsymbol{\Lambda}(\cdot)$ is Lipschitz continuous in $\mathcal{N}$. The latter holds because $\boldsymbol{\Lambda}(\cdot)$ has bounded Jacobean $\boldsymbol{R}^*$ at $\boldsymbol{\tau}^*$, and the Jacobean is continuous everywhere. Therefore, the state-dependent arrival rate of the process $\boldsymbol{Q}^{\kappa,n}$ is Lipschitz continuous over its domain. Hence, we can invoke Theorem 7.2 in Mandelbaum et al. (1998b) and show that

$$\{\boldsymbol{Q}^{\kappa,n}(t) | 0 \leq t \leq T\} \Rightarrow \{\boldsymbol{Y}^{\kappa}(t) | 0 \leq t \leq T\}. \tag{53}$$

Finally, for all bounded, continuous real-valued function f with domain $D([0,T),\mathbb{R}^J)$, when $\kappa \to \infty$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} |\mathbb{E}f(\boldsymbol{Q}^n) - \mathbb{E}f(\boldsymbol{Y})| \\
&\leq \limsup_{n\to\infty} |\mathbb{E}f(\boldsymbol{Q}^n) - \mathbb{E}f(\boldsymbol{Q}^{\kappa,n})| + \limsup_{n\to\infty} |\mathbb{E}f(\boldsymbol{Q}^{\kappa,n}) - \mathbb{E}f(\boldsymbol{Y}^{\kappa,n})| + |\mathbb{E}f(\boldsymbol{Y}^{\kappa}) - \mathbb{E}f(\boldsymbol{Y})| \\
&\leq \limsup_{n\to\infty} 2\overline{f}\,\Pr(\|\boldsymbol{Q}^n - \boldsymbol{Q}^{\kappa,n}\|_T \neq 0) + 0 + |\mathbb{E}f(\boldsymbol{Y}^{\kappa}) - \mathbb{E}f(\boldsymbol{Y})| \\
&\to 0
\end{aligned} \tag{54}$$

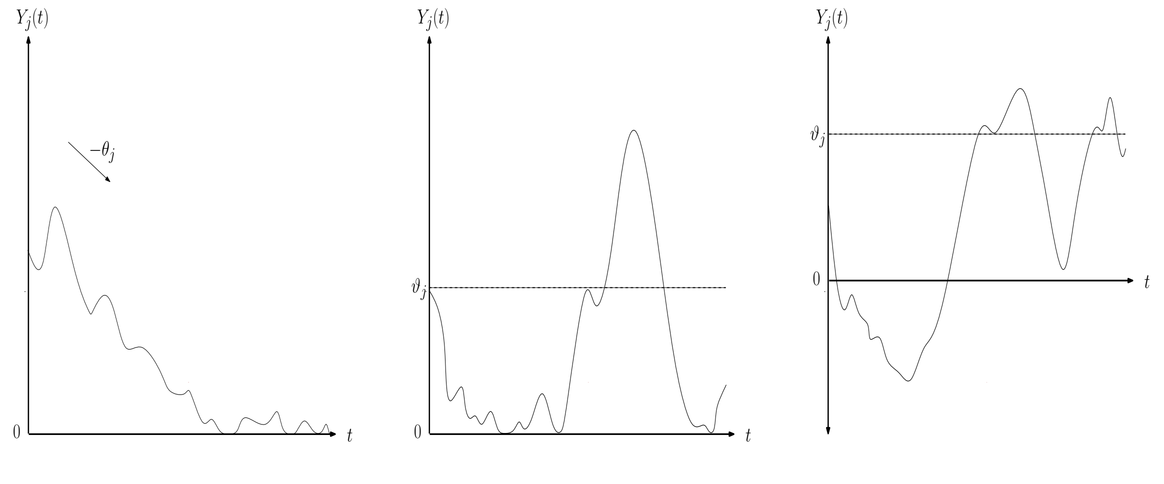
where $\overline{f}$ represents an upper bound for $|f|$, $\limsup_{n\to\infty} |\mathbb{E}f(\boldsymbol{Q}^{\kappa,n}) - \mathbb{E}f(\boldsymbol{Y}^{\kappa,n})| = 0$ follows from Equation (53), $\limsup_{n\to\infty} 2\overline{f}\,\Pr(\|\boldsymbol{Q}^n - \boldsymbol{Q}^{\kappa,n}\|_T \neq 0) \to 0$ follows from Lemma 3, and $|\mathbb{E}f(\boldsymbol{Y}^{\kappa}) - \mathbb{E}f(\boldsymbol{Y})| \to 0$ follows from bounded convergence and the continuous mapping theorem. Equation (54) implies that $\boldsymbol{Q}^n \Rightarrow \boldsymbol{Y}$. ■

Perhaps the most useful characterization of a stochastic process is its stationary distribution. The diffusion limit process $\boldsymbol{Y}$ is an RMOU and falls into the category of multi-dimensional reflected diffusion processes, the stationary distribution of which has been studied in (Dieker and Gao, 2013; Kang and Ramanan, 2014). Based on the results of Kang and Ramanan (2014), we can derive a closed-form characterization of the stationary distribution of $\boldsymbol{Y}$ under additional assumptions that the Jacobean is symmetric and all service providers have the same coefficient of variation.

Proposition 3 *(Stationary Distribution of the Diffusion Limit) The RMOU process $\boldsymbol{Y}$ has a unique stationary distribution. Furthermore, if the $\boldsymbol{R}^*$ is symmetric, and $\omega_j \equiv \omega_1$ for all $j \in \mathcal{J}$, then the stationary distribution of $\boldsymbol{Y}$ defined in Theorem 4 is a truncated multivariate Gaussian distribution, and its density has a closed form*

$$\pi_{\boldsymbol{Y}}(\boldsymbol{z}) = \begin{cases} \frac{\pi(\boldsymbol{z})}{\int_\Omega \pi(\boldsymbol{z})d\boldsymbol{z}} & \textit{if } \boldsymbol{z} \in \Omega, \\ 0 & \textit{otherwise.} \end{cases} \tag{55}$$

where $\pi(\boldsymbol{z})$ is the density function of a multivariate Gaussian distribution with mean $\boldsymbol{\vartheta} + \mathrm{Diag}\,(\boldsymbol{\mu})(\boldsymbol{R}^*)^{-1}\boldsymbol{\theta}$ and covariance matrix $-\frac{1}{2}(1+\omega_1^2)\mathrm{Diag}\,(\boldsymbol{\mu})(\boldsymbol{R}^*)^{-1}\mathrm{Diag}\,(\boldsymbol{\mu})$.



(a)When queue j is slightly under-demand, Y_j tends to move toward the virtual equilibrium $\vartheta_j = 0$ at a constant downward drift rate θ_j. Meanwhile, 0 is a reflection barrier for Y_j.

(b)When queue j is slightly over-demand (or balanced), Y_j oscillates around the virtual equilibrium ϑ_j and is subject to a reflection barrier at 0.

(c)When queue j is largely over-demand, Y_j oscillates around the virtual equilibrium ϑ_j in an unbounded domain.

Figure 5 Typical sample paths of Y_j in the cases of $j \in \mathcal{J}^-, \mathcal{J}^+, \mathcal{J}^{++}$.

Note that the symmetry of $\boldsymbol{R}^*$ and the assumption $\omega_j \equiv \omega_1$ is indispensable for the current methodology to work, i.e., using the result of Kang and Ramanan (2014). Otherwise we lose symmetry of $\boldsymbol{\Sigma}^{-2}\boldsymbol{R}^*\text{Diag}\,(\boldsymbol{\mu}^{-1})$ and cannot apply the result of Kang and Ramanan (2014). Proposition 3 follows from Example 3.10, Claim 1 of Kang and Ramanan (2014). A detailed proof is provided in Appendix G.

Remark 8 *The multivariate Gaussian steady-state distribution provides the system manager with some practical insights. Since the covariance matrix of such a distribution is proportional to the inverse of the Jacobean $(\boldsymbol{R}^*)^{-1}$, the spread of the distribution is decreasing in the scale of $\boldsymbol{R}^*$. Thus if one wishes to reduce the variability of the queue-length process of the PQCDA, one may consider increasing the scale of $\boldsymbol{R}^*$, which depends on customers' delay sensitivity. Roughly, if customers are more sensitive to the non-zero waiting times (so a larger c_ξ), then $\boldsymbol{R}^*$ will have a larger scale which leads to a lower spread of the multivariate Gaussian distribution. Thus the diffusion limit process will be more concentrated at its center. Such a reduction in queue length variability will load the multi-queue service system in a more balanced way which reduces the idle times of all servers and increases the system throughput. Therefore, the system manager has an incentive to exert effort to persuade customers to actively use the queue-length information. As a result, the customer's delay sensitivity can be increased so that the pooling effect of PQCDA can be enhanced and the system efficiency can be improved.*

Proposition 3 presented above states that the stationary distribution of the limiting process $\boldsymbol{Y}$, denoted by $\boldsymbol{\pi}$, is truncated multivariate Gaussian and has a closed-form density function. In the n^{th} PQCDA, if we let $b_j^n(t)$, $j = 1, 2, \ldots, J$ denote the remaining service time for the customer currently being served by the j^{th} service provider, and define $\boldsymbol{b}^n(t) := (b_j^n(t))$. Then $\boldsymbol{\Xi^n}(t) := (\boldsymbol{Q}^n(t), \boldsymbol{b}^n(t))$ is a Markov process and can be proved to have a stationary distribution. Let $\boldsymbol{\pi}^n$ denote the projection of the stationary distribution onto $\boldsymbol{Q}^n$. Then we can prove that $\boldsymbol{\pi}^n$ weakly converges to $\boldsymbol{\pi}$ when n approaches infinity. This result is also termed as *interchange of limits* and illustrated in Figure 6.

The interchange of limits was proved when Ξ^n is the Markov process in a generalized Jackson network by Gamarnik and Zeevi (2006). We adopt their machinery and show that the interchange of limits holds for the PQCDA. The queueing network considered by Gamarnik and Zeevi (2006) assume constant arrival and service rates, while the arrival rates in our model are state-dependent and non-Lipschitz. Therefore, the adoption of their methods is not trivial and must exploit the choice-driven property. Specifically, the choice-driven property is used to prove that a *Lyapunov function* can be constructed so that its exponential has bounded expectation.

Formally, a function $V:\Omega\to\mathbb{R}_+$ is said to be a Lyapunov function with drift size parameter $-\gamma<0$ and drift time parameter $t_0>0$ and exception parameter κ for a Markov process Ξ if

$$\sup_{\Xi(0)\in\Omega:\ V(\Xi(0))>\kappa}\{\mathbb{E}_{\boldsymbol{\Xi(0)}}V(\Xi(t_0))-V(\Xi(0))\}\le-\gamma. \tag{56}$$

For each n, define

$$\begin{aligned}L_1(u,t,n) &:= \sup_{\Xi^n(0)\in\Omega}\mathbb{E}[\exp(u(V(\Xi^n(t))-V(\Xi^n(0))))|\Xi^n(0)]\\ L_2(u,t,n) &:= \sup_{\Xi^n(0)\in\Omega}\mathbb{E}[(V(\Xi^n(t))-V(\Xi^n(0)))^2\exp(u(V(\Xi^n(t))-V(\Xi^n(0)))^+)|\Xi^n(0)]\end{aligned} \tag{57}$$

for any $u>0$, $t\ge0$. We then have the following proposition.

Proposition 4 *Let $V(\Xi^n(t)):=\|\boldsymbol{Q}^n(t)\|^{\boldsymbol{\mu}^{-1}}$. Then for sufficiently large n, $V(\cdot)$ is a Lyapunov function with drift size parameter -1, drift time parameter t_0, and exception parameter κ for some $\kappa,t_0>0$. In addition, there exists u_0 such that*

$$\begin{aligned}\limsup_{n\to\infty}L_1(u_0,t_0,n) &< \infty\\ \limsup_{n\to\infty}L_2(u_0,t_0,n) &< \infty\end{aligned} \tag{58}$$

The above proposition is in analogue to Proposition 3 in Gamarnik and Zeevi (2006), but deals with the PQCDA case in which the drift function is not Lipschitz. Note that we have used different notations from those used in Gamarnik and Zeevi (2006): our $\boldsymbol{Q}^n(t)$ corresponds to the notation "$\frac{1}{\sqrt{n}}\boldsymbol{Q}^n(nt)$" in their paper. Because we have used a different scale, the bound we derived with respect to the $\|\cdot\|_{t_0}$ norm is exactly the bound derived in their paper the interval $[0,nt_0]$. A complete proof for Proposition 4 is provided in Appendix H.

$$\begin{tikzcd}\mathbf{Q}^n(t) \arrow[r, "t\to\infty"] \arrow[d, "n\to\infty"] & \mathbf{Q}^n(\infty) \arrow[d, "n\to\infty"] \\ \mathbf{Y}(t) \arrow[r, "t\to\infty"] & \mathbf{Y}(\infty)\end{tikzcd}$$

Figure 6 The interchange-of-limit result implies that the steady-State distribution of $\boldsymbol{Y}^n$, $\boldsymbol{\pi}$, can be approximated by $\boldsymbol{\pi}^n$, the projection of the steady-state distribution of Ξ^n onto the subspace of $\boldsymbol{Q}^n$.

Theorem 5 *(Interchange of Limit) The sequence of stationary distributions, π^n, weakly converges to π.*

The main idea of the proof is to construct a Lyapunov function with the properties given in Proposition 4. Those properties allow us to prove uniform tightness of the sequences $(\boldsymbol{\pi}^n)$, which then yields the existence of a limiting distribution $\hat{\boldsymbol{\pi}}$. The interchange of limits can then be proved by arguing that any such $\hat{\boldsymbol{\pi}}$ must coincide with the unique stationary distribution of $\boldsymbol{Y}$, $\boldsymbol{\pi}$. A complete proof is provided in Appendix I.

8. PQCDA with Reneging Customers

The previous results on PQCDA without reneging customers can be extended to the case when customers may renege (or abandon) after an exponentially distributed time before getting served. Note that in most past studies on the queues with customer choice, the reneging feature was not considered due to the reason that a customer's decision to join was made based on the expected service utility. We incorporate reneging, as it is a feature in our motif dating examples. For example, in health care settings, death or unexpected changes in medical conditions may lead to abandonment of the service by patients. Since the analysis with reneging is similar to the one in earlier sections, we only elaborate the results where the technical differences are significant.

We assume that customers renege after an exponentially distributed period with mean of $1/d$. When the system is Markovian (the inter-arrival times, reneging times, and service times are all exponentially distributed), the following expression given in Zenios (1999) can be used to compute the expected waiting time

$$\tau_j = \frac{1}{d}\log(1 + \frac{X_j d}{\mu_j}). \tag{59}$$

We assume that all customers use (59) to compute their expected waiting time, and choose a queue which maximizes their payoff $U_{\xi,j}$ as given in (1), which leads to state-dependent arrival rate function $\boldsymbol{\Lambda}(\boldsymbol{\tau})$. Because our proof for Proposition 1 does not rely on the functional form of τ_j with respect to X_j, the proof can be adapted to establishing the choice-driven property of the arrival rate function in the presence of reneging customers.

Corollary 1 *With the customer choice model defined in Section 3, even if customers renege after an exponentially distributed time with mean* $1/d$ *before service, the arrival rate function still satisfies the CD property, and its Jacobean is symmetric.*

Remark 9 *With reneging, the PQCDA is always stable. So the stability condition* (9) *is no longer necessary.*

We next prove that the fluid process in a PQCDA with customer reneging converges to the equilibrium state, which is the unique solution to an NCP with a slightly different formulation compared to the non-reneging case.

Theorem 6 *The equilibrium state of the fluid limit process in PQCDA with reneging is the unique solution to the following Nonlinear Complementary Problem (NCP).*

$$\textit{NCP} \qquad \begin{aligned} Z_j := \mu_j \exp(\tau_j d) - p_j(\boldsymbol{\tau}) &\geq 0, \quad \textit{for } j = 1, \dots, J. \\ \tau_j &\geq 0, \quad \textit{for } j = 1, \dots, J. \\ \tau_j Z_j &= 0, \quad \textit{for } j = 1, \dots, J. \end{aligned} \tag{60}$$

Moreover, if we use $\boldsymbol{\tau}(t)$ *to denote the waiting-time vector in a fluid model, then for any given* $\boldsymbol{\tau}(0) \geq 0$, $\boldsymbol{\tau}(t) \to \boldsymbol{\tau}^*$.

Proof. By defining $\hat{p}_j := \mu_j \exp(\tau_j d) - p_j(\boldsymbol{\tau})$, the above NCP can be rewritten into a similar form as in(40) by replacing the arrival function $\boldsymbol{\Lambda}(\cdot)$ with $\hat{\boldsymbol{\Lambda}}(\cdot) := (\hat{p}_j(\cdot))_{j=1,\ldots,J}$. Note that the Jacobian for $\hat{\boldsymbol{\Lambda}}(\boldsymbol{\tau})$ has the form $\hat{R} = \sigma(\boldsymbol{\tau}) + R$, where R is the Jacobian of $p(\boldsymbol{\tau})$ and is a symmetric negative definite matrix by Corollary 1, and $\sigma(\boldsymbol{\tau})$ is a diagonal matrix with the j^{th} entry $\sigma_{jj}(\boldsymbol{\tau}) = \mu_j d \exp(\boldsymbol{\tau}_j d) > 0$. Because of the extra term $\sigma(\boldsymbol{\tau})$, we are now able to prove that $\hat{p}_j$ satisfies the uniform P-property, i.e.,

$$\text{Uniform P-Property: } \forall \boldsymbol{\tau}^1, \boldsymbol{\tau}^2 \in \mathbb{R}_+^J,\ \boldsymbol{\tau}^1 \neq \boldsymbol{\tau}^2,\ \min_{j=1}^{J}\ (\tau_j^1 - \tau_j^2)(\hat{p}_j(\boldsymbol{\tau}^1) - \hat{p}_j(\boldsymbol{\tau}^2)) < c\|\boldsymbol{\tau}^1 - \boldsymbol{\tau}^2\|^2, \quad (61)$$

with $c > d \max_j \mu_j > 0$. Thus, the classical theorem by Cottle (1966) implies the existence of a unique solution to the NCP (60).

To prove $\boldsymbol{\tau}(t) \to \boldsymbol{\tau}^*$, we define $\overline{\Delta}\boldsymbol{\tau}(t) = \max_j(\tau_j(t) - \tau_j^*)$, and $\underline{\Delta}\boldsymbol{\tau}(t) = \min_j(\tau_j(t) - \tau_j^*)$. We want to prove that $\overline{\Delta}\boldsymbol{\tau}'(t) \leq \kappa(\delta)$ for some constant $\kappa(\delta) > 0$ whenever $\overline{\Delta}\boldsymbol{\tau}(t) \geq \delta$. Without loss of generality, assume that $\tau_j(t) - \tau_j^* = \overline{\Delta}\boldsymbol{\tau}'(t)$, then $\tau_j(t) > 0$ and (59) imply that

$$\tau_j'(t) = \frac{p_j(\boldsymbol{\tau}) - \mu_j}{X_j(t)d + \mu_j} \leq \frac{p_j - \mu_j}{p_j(\boldsymbol{\tau}^*)}, \quad (62)$$

where the inequality follows from the NCP constraint $Z_j = \mu_j(\exp(\tau_j d)) - p_j(\boldsymbol{\tau}^*) = \mu_j + X_j^* d - p_j(\boldsymbol{\tau}^*) \geq 0$.

The rest of the proof resembles the proof of Theorem 3, i.e., we prove facts (1) and (2) and show that $\overline{\Delta}\boldsymbol{\tau}'(t) \leq -\kappa(\delta)$. We then use the similar argument to show that $\underline{\Delta}\boldsymbol{\tau}'(t) \geq \kappa(\delta)$ whenever $\underline{\Delta}\boldsymbol{\tau}(t) \leq -\delta$ and prove $\boldsymbol{\tau}(t) \to \boldsymbol{\tau}^*$. ■

The proof of the convergence to the diffusion limit is a simple extension of Theorem 4 by including an extra term $-dI$ in the drift matrix as a result of reneging. We summarize the result below and the notations follow from the definitions in the previous sections.

We next study the diffusion approximation for $\boldsymbol{Q}^n(\cdot) := (Q_j^n(\cdot))$, where $Q_j^n(\cdot)$ is the scaled queue-length process in the n^{th} PQCDA with its expression given in Equation (45). As before we partition the index set of queues into four subsets $\mathcal{J}^{--}$, $\mathcal{J}^{-}$, $\mathcal{J}^{+}$, and $\mathcal{J}^{++}$ according to (43) and assume $\mathcal{J}^{--} = \emptyset$. We redefine $\boldsymbol{\theta}$ and $\boldsymbol{\vartheta}$ as follows,

$$\theta_j = \begin{cases} \lim_{n\to\infty} \sqrt{n}(\mu_j^n - p_j(\boldsymbol{\tau}^{n,*})) & \text{if } j \in \mathcal{J}^- \\ 0 & \text{otherwise,} \end{cases} \quad \vartheta_j = \begin{cases} \lim_{n\to\infty} \sqrt{n}(X_j^{n,*} - X_j^*) & \text{if } j \in \mathcal{J}^+ \cup \mathcal{J}^{++} \\ 0 & \text{otherwise,} \end{cases} \quad (63)$$

where $X_j^{n,*} = \frac{\mu_j^n}{d}(\exp(d\tau_j^{n,*}) - 1)$ represents the queue length when the expected waiting time is $\tau_j^{n,*}$. The scaled-queue length process $\boldsymbol{Q}^n(\cdot)$ and its diffusion limit $\boldsymbol{Y}(\cdot)$ thus reside in the following domain,

$$\Omega = [0, +\infty)^{J^- \cup J^+} \otimes (-\infty, +\infty)^{J^{++}}. \quad (64)$$

The next Corollary, which is analogous to Theorem 4, states that $\boldsymbol{Q}^n$ converges to a J-dimensional diffusion process $\boldsymbol{Y}$ which is the solution to the following stochastic-differential-equation,

$$\boldsymbol{Y}(t) = \int_0^t \big((\boldsymbol{R}^* \mathrm{Diag}\,(((\boldsymbol{e} + \boldsymbol{\tau}^* d) \circ \boldsymbol{\mu})^{-1}) - dI)(\boldsymbol{Y}(s) - \boldsymbol{\vartheta}) - \boldsymbol{\theta}\big)\, ds + \int_0^t \boldsymbol{\Sigma}^R d\boldsymbol{B}(s) + \boldsymbol{L}(t), \quad (65)$$

where I is an J-by-J identity matrix, $\boldsymbol{\Sigma}^R$ is a J-by-J diagonal matrix with $\sqrt{(\omega_j^2 + \exp(\tau_j^* d))\mu_j}$ as its j^{th} diagonal entry, $\boldsymbol{B}(t)$ is a J-dimensional Brownian motion, and $\boldsymbol{L}(t)$ is a J-dimensional minimal non-decreasing process which makes $Y_j(t) \geq 0$ for all $j \in \mathcal{J}^- \cup \mathcal{J}^+$.

Corollary 2 *Suppose $\boldsymbol{Q}^n(0) \Rightarrow \boldsymbol{Y}(0)$ and $\mathbb{E}\|\boldsymbol{Y}(0)\| < \infty$. Then we have*

$$\boldsymbol{Q}^n \Rightarrow \boldsymbol{Y}. \quad (66)$$

The proof for Corollary 2 is provided in Appendix K.

9. Conclusions and Future Research

Our paper is the first to apply heavy traffic approximations to the general PQCDA problem and derive properties of its steady sate. As mentioned, we not only make theoretical contributions but also address some of the managerial issues of interest that arise in practice. For example, our results can directly help practitioners implement system evaluation metrics for controlling these types of stochastic systems. Future work can evaluate the value of information by comparing the social welfare achieved in PQCDA versus a parallel-queue system without waiting time announcements. Another important question would be to evaluate the discrepancy between the social welfare optimization to that of a customers self-interest maximization in a PQCDA. The results can also be applied to evaluate the performance of PQCDAs under different staffing policies, which we were unable to do for the border-crossing queues due to the lack of data on customer balking.

Our analytical framework can be extended to a PQCDA in which all customers renege after an identically and exponentially distributed random period. However, if reneging is endogenous (state-dependent), then the problem is known to be hard (Ata and Peng, 2018). Also, in some situations, waiting customers may abandon the current queue and join a different queue. Usually, when a customer abandons the current queue, she has to lose her priority in that queue and has to wait at the end of the new queue. Such a switching behavior is equivalent to the event that a customer reneges in one queue and a new customer joins another queue. Our conjecture is that this will not change the behavior of the PQCDA and thus will not affect the asymptotic characterization significantly. Relaxing some of the technical assumptions, such as Poisson arrival and exponential reneging times, can be an interesting but challenging and is left for future research.

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Appendix A: Proof of Proposition 1

Proof. We first prove that the Jacobean of the arrival rate function exists and is continuous at all $\boldsymbol{\tau} \notin \mathcal{K}^J$.

For all $j \neq i$ and $i, j \neq 0$, if the partial derivative $\frac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i}$ exists, then it must equal to the following limit

$$\lim_{t \to 0} \frac{1}{t}(p_j(\boldsymbol{\tau} + t\boldsymbol{e_i}) - p_j(\boldsymbol{\tau})). \tag{67}$$

Note that $\boldsymbol{\tau} + t\boldsymbol{e_i}$ and $\boldsymbol{\tau}$ differs only in the i^{th} component. Thus, if a customer of type ξ chooses to join queue j at $\boldsymbol{\tau} + t\boldsymbol{e_i}$, but not to join queue j at $\boldsymbol{\tau}$, then he must have chosen queue i at

$\boldsymbol{\tau}$. Because his utility of joining other queues is not changed. Those customers must have their parameters $(\boldsymbol{u}_\xi, c_\xi)$ contained in the set $S^1 \cap S^2(t)$, where

$$\begin{aligned} S^1 &:= \left\{ (\boldsymbol{u}, c) \,|\, \begin{matrix} u_j - c\tau_j > \max\{0, u_k - c\tau_k,\ k \neq i, j\} \\ u_i - c\tau_i > \max\{0, u_k - c\tau_k,\ k \neq i, j\} \end{matrix} \right\} \\ S^2(t) &:= \left\{ (\boldsymbol{u}, c) | \, c(\tau_i - \tau_j) \leq u_i - u_j < c(\tau_i - \tau_j + t) \right\} \end{aligned} \tag{68}$$

Intuitively, $\xi \in S^1$ if queue i and queue j are the top two choices of customer ξ; $\xi \in S^2$ if the expected utility of queue i and queue j are so close that a small change of τ_i would alter his choice. The probability for $\xi \in S^1 \cap S^2(t)$ is thus exactly the difference $p_j(\boldsymbol{\tau} + t\boldsymbol{e_i}) - p_j(\boldsymbol{\tau})$.

If $\tau_i \neq \tau_j$, the limit (67) can be calculated as

$$\begin{aligned} &\lim_{t\to 0} \frac{1}{t}(p_j(\boldsymbol{\tau} + t\boldsymbol{e_i}) - p_j(\boldsymbol{\tau})) \\ =& \lim_{t\to 0} \frac{1}{t} \Pr\left((\boldsymbol{u}, c) \in S(t)\right) \\ =& \lim_{t\to 0} \frac{1}{t} \int_{(\boldsymbol{u},c)\in S^2(t)} I((\boldsymbol{u}, c) \in S_1) f(\boldsymbol{u}, c)\, d\boldsymbol{u} dc \\ =& \lim_{t\to 0} \int [\frac{1}{t} \int_{\frac{u_i - u_j}{\tau_i - \tau_j + t}}^{\frac{u_i - u_j}{\tau_i - \tau_j}} f_{c|\boldsymbol{u}}(c) I((\boldsymbol{u}, c) \in S_1)\, dc] f(\boldsymbol{u})\, d\boldsymbol{u} \\ =& \int \lim_{t\to 0} [\frac{1}{t} \int_{\frac{u_i - u_j}{\tau_i - \tau_j + t}}^{\frac{u_i - u_j}{\tau_i - \tau_j}} f_{c|\boldsymbol{u}}(c) I((\boldsymbol{u}, c) \in S_1)\, dc] f(\boldsymbol{u})\, d\boldsymbol{u} \qquad (69) \\ =& \int I((\boldsymbol{u}, \frac{u_i - u_j}{\tau_i - \tau_j}) \in S_1) f(\boldsymbol{u}, \frac{u_i - u_j}{\tau_i - \tau_j})\, d\boldsymbol{u}. \qquad (70) \end{aligned}$$

Equality (69) is due to dominated convergence. To see that, note that the term inside $[\cdot]$ has the following limit

$$\lim_{t\to 0} [\frac{1}{t} \int_{\frac{u_i - u_j}{\tau_i - \tau_j + t}}^{\frac{u_i - u_j}{\tau_i - \tau_j}} f_{c|\boldsymbol{u}}(c) I((\boldsymbol{u}, c) \in S_1)\, dc] = f_{c|\boldsymbol{u}}(\frac{u_i - u_j}{\tau_i - \tau_j})\, I((\boldsymbol{u}, \frac{u_i - u_j}{\tau_i - \tau_j}) \in S_1). \tag{71}$$

Thus, for sufficiently small t, the term inside $[\cdot]$ is upper bounded by $2f_{c|\boldsymbol{u}}(\frac{u_i - u_j}{\tau_i - \tau_j}) I((\boldsymbol{u}, \frac{u_i - u_j}{\tau_i - \tau_j}) \in S_1)$, whose integral with respect to $\boldsymbol{u}$ is upper bounded by the marginal density $2f_c(\frac{u_i - u_j}{\tau_i - \tau_j})$.

Therefore, if $\tau_i \neq \tau_j$, the partial derivative $\frac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i}$, as the limit of $\frac{1}{t}(p_j(\boldsymbol{\tau} + t\boldsymbol{e_i}) - p_j(\boldsymbol{\tau}))$, exists and has the following expression,

$$\frac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i} = \int I((\boldsymbol{u}, \frac{u_i - u_j}{\tau_i - \tau_j}) \in S_1) f(\boldsymbol{u}, \frac{u_i - u_j}{\tau_i - \tau_j})\, d\boldsymbol{u}. \tag{72}$$

Since the RHS of above equation is a continuous function of τ_i and τ_j when $\tau_i \neq \tau_j$, the partial derivative $\frac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i}$ must be continuous at all $\boldsymbol{\tau} \notin \mathcal{K}^J$ (so $\tau_i \neq \tau_j$).

The above argument proves that if $j \neq i$, then $\frac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i}$ exists and is continuous for all $\boldsymbol{\tau} \notin \mathcal{K}^J$. It remains to prove the above property of $\frac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i}$ for the $j = i$ case. Because $\sum_{i=0}^J p_j(\boldsymbol{\tau}) \equiv 1$, we know that

$$\tfrac{\partial p_i(\boldsymbol{\tau})}{\partial \tau_i} = -\textstyle\sum_{j\neq i,\, j=0,1,\ldots,J} \tfrac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i} \tag{73}$$

Note that the summation at the RHS consists of $\frac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i}$ for all $j \neq i$ (including $j = 0$). $p_0(\boldsymbol{\tau})$ represents the proportion of customers who choose to balk, or equivalently, to join a queue indexed by 0 with expected waiting time $\tau_0 = 0$ and service utility $u_0 = 0$. Thus, using the previous argument for the $i \neq j$ case, we can show that $\frac{\partial p_0(\boldsymbol{\tau})}{\partial \tau_i}$ exists and is continuous for all $\boldsymbol{\tau} \notin \mathcal{K}^J$. Because for all $j \neq i$ (including $j = 0$), $\frac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i}$ exists and is continuous at all $\boldsymbol{\tau} \notin \mathcal{K}^J$, Equation (73) implies that $\frac{\partial p_i(\boldsymbol{\tau})}{\partial \tau_i}$ exists and is continuous except at all $\boldsymbol{\tau} \notin \mathcal{K}^J$.

So far, we have proved that the arrival rate function $p_j(\boldsymbol{\tau})$ has continuous derivatives everywhere except at points in $\mathcal{K}^J$. Next we show that even at points in $\mathcal{K}^J$, $p_j(\boldsymbol{\tau})$ is continuous, though it may not have finite derivatives. Thus, $p_j(\boldsymbol{\tau})$ is absolute continuous. Formally,

$$\begin{aligned}
&\lim_{t\to 0} p_j(\boldsymbol{\tau} + t\boldsymbol{e_i}) - p_j(\boldsymbol{\tau}) \;=\; \lim_{t\to 0} \Pr\left((\boldsymbol{u}, c) \in S(t)\right) \\
&= \lim_{t\to 0} \int \int [\int_0^{ct} f_{u_i|\boldsymbol{u}_{-i},c}(u_j + x) I((\boldsymbol{u}, c) \in S_1)\, du_i] f_{\boldsymbol{u}_{-i},c}(\boldsymbol{u}_{-i}, c)\, dc\, d\boldsymbol{u}_{-i} && (74) \\
&= 0 && (75)
\end{aligned}$$

where equality (74) follows from Equality (75) follows from that $\lim_{t\to 0} \int_0^{ct} f_{u_i|\boldsymbol{u}_{-i},c}(u_j + x) I((\boldsymbol{u}, c) \in S_1)\, du_i = 0$. We may repeatedly apply the above logic for each coordinate $i \neq j$ and establish continuity of $p_j(\boldsymbol{\tau})$ at points in $\mathcal{K}^J$.

We next prove (CD-a)-(CD-c).

(CD-a): Suppose $\tau_k^2 > \tau_k^1$, and $\tau_l^2 = \tau_l^1$ for $j \neq k$. For a customer indexed by ξ, if his choice is queue $j \neq k$, then

$$u_k - c\tau_k^2 < u_k - c\tau_k^1 \leq u_j - c\tau_j^1 = u_j - c\tau_j^2, \tag{76}$$

where the first inequality is due to $\tau_k^2 > \tau_k^1$, the second inequality follows from the fact that the customer's optimal choice is queue j instead of queue k, and the last equality follows since $\tau_j^1 = \tau_j^2$. Therefore, if a customer's initial choice is queue j, then his choice remains the same when the waiting-time vector is changed from $\boldsymbol{\tau}^1$ to $\boldsymbol{\tau}^2$. We thus deduce that $p_j(\boldsymbol{\tau})$ is non-decreasing in τ_k.

(CD-b): Note that $p_j(\boldsymbol{\tau})$ must be non-increasing with τ_j as a result of (CD-a) and $\sum_{k=0}^{J} p_k = 1$. So it suffices to prove $p_j(\boldsymbol{\tau})$ is strictly decreasing when $\boldsymbol{\tau}^1$ has been replaced by $\boldsymbol{\tau}^2$, where $\tau_j^2 > \tau_j^1$ but $\tau_k^2 = \tau_k^1$ for $k \neq j$. A customer ξ will choose to join queue j given expected waiting-times vector $\boldsymbol{\tau}^1$, but not join queue j when the waiting-time vector is changed to $\boldsymbol{\tau}^2$, if and only if

$$(\boldsymbol{u}_\xi, c_\xi) \in \{(\boldsymbol{u}, c)| \begin{matrix} u_j - c\tau_j^1 > \max\{0, u_k - c\tau_k^1,\ k \neq j\} \\ u_j - c\tau_j^2 < \max\{0, u_k - c\tau_k^2,\ k \neq j\} \end{matrix}\} \tag{77}$$

Because the parameter c has positive conditional pdf $f_{c|\boldsymbol{u}}$ over $\mathbb{R}_+$, the above set must have a positive probability mass. Therefore, a positive proportion of customers must switch to queues other than j when the waiting time of queue j has been increased from τ_j^1 to τ_j^2. Therefore, $p_j(\boldsymbol{\tau})$ is strictly decreasing in τ_j.

(CD-c): Given $\boldsymbol{\tau}^2 := \boldsymbol{\tau}^1 + t\boldsymbol{e}$, the linear form of $U_{\xi,j}$ implies that if $U_{\xi,j} \geq U_{\xi,k}$ for all $k \neq j$ (including $k = 0$) at $\boldsymbol{\tau}^2$, then the same inequalities must hold at $\boldsymbol{\tau}^1$. Therefore, we deduce that $p_j(\boldsymbol{\tau}^1) \geq p_j(\boldsymbol{\tau}^2)$ for all $j \neq 0$. To prove the strict inequality in (12), we notice that a customer of type ξ joins some queue at $\boldsymbol{\tau}$, but balks at $\boldsymbol{\tau}^2$ if

$$(\boldsymbol{u}, c) \in \left\{ (\boldsymbol{u}, c) \mid \begin{matrix} 0 < \max\{u_k - c\tau_k^1,\ k = 1, \ldots, J\} \\ 0 > \max\{u_k - c(\tau_k^2 + t),\ k = 1, \ldots, J\} \end{matrix} \right\}. \tag{78}$$

Because the parameter c has positive conditional pdf $f_{c|\boldsymbol{u}}$ over $\mathbb{R}_+$, the above set must have a positive probability mass, so the strict inequality (12) is proved, which implies row strict diagonally dominance of the Jacobean matrix. Inequality (13) and the column strict diagonally dominance follow from symmetry of the Jacobean matrix, a result that will be proved in the end of this proof.

We next prove the stability condition (9). A customer will joint queue j only if $u_j - c\tau > 0$. Therefore, when $\tau_j \to \infty$,

$$p_j(\boldsymbol{\tau}) \le \Pr(u_j - c\tau > 0) = \int [\int_0^{u_j/\tau_j} f_{c|\boldsymbol{u}}(c)\, dc]\, f_{\boldsymbol{u}}(\boldsymbol{u}) d\boldsymbol{u} \to 0. \tag{79}$$

where the convergence follows from $u_j/\tau_j \to 0$ and our assumption that $f_{c|\boldsymbol{u}}(\cdot)$ is bounded. Equation (79) leads to (9).

Finally, we prove that the Jacobean matrix is symmetric whenever it exists. Equation (69) implies that

$$\begin{aligned} \tfrac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i} &= \lim_{t\to 0} \tfrac{1}{t}(p_j(\boldsymbol{\tau} + t\boldsymbol{e_i}) - p_j(\boldsymbol{\tau})) \\ &= \lim_{t\to 0} \tfrac{1}{t} \Pr\left(\left\{ (\boldsymbol{u}, c) | \begin{array}{rl} u_j - c\tau_j &> \max\{0, u_k - c\tau_k,\ k \ne i, j\} \\ u_i - c\tau_i &> \max\{0, u_k - c\tau_k,\ k \ne i, j\} \\ u_j - c\tau_j &> u_i - c(\tau_i + t) \\ u_j - c\tau_j &< u_i - c\tau_i \end{array} \right\} \right) \end{aligned} \tag{80}$$

Similarly,

$$\begin{aligned} \tfrac{\partial p_i(\boldsymbol{\tau})}{\partial \tau_j} &= \lim_{t\to 0} \tfrac{1}{t}(p_i(\boldsymbol{\tau}) - p_i(\boldsymbol{\tau} - t\boldsymbol{e_j})) \\ &= \lim_{t\to 0} \tfrac{1}{t} \Pr\left(\left\{ (\boldsymbol{\alpha}, c, \boldsymbol{\epsilon}) | \begin{array}{rl} u_j - c\tau_j &> \max\{0, u_k - c\tau_k,\ k \ne i, j\} \\ u_i - c\tau_i &> \max\{0, u_k - c\tau_k,\ k \ne i, j\} \\ u_j - c(\tau_j - t) &> u_i - c\tau_i \\ u_j - c\tau_j &< u_i - c\tau_i \end{array} \right\} \right) \end{aligned} \tag{81}$$

Notice that the set at the RHS of Equation (80) and (81) are identical. The intuition is that it is the same group of customers who will switch to queue j, when either τ_j has been decreased by t, or τ_j has been increased by t. We thus have $\partial p_j(\boldsymbol{\tau})/\partial \tau_i = \partial p_i(\boldsymbol{\tau})/\partial \tau_j$ and symmetry is proved. ■

The above proof also leads to an example that the arrival rate function $\boldsymbol{\Lambda}(\cdot)$ does not have to be (even locally) Lipschitz continuous. In particular, its partial derivative may be infinite at points in K^J. Let $\boldsymbol{u}_{-i}$ denote the vector obtained by removing the i^{th} entry from $\boldsymbol{u}$. Equation (74) then implies that

$$\begin{aligned} &\liminf_{t\to 0} \frac{1}{t}(p_j(\boldsymbol{\tau} + t\boldsymbol{e_i}) - p_j(\boldsymbol{\tau})) \\ &= \liminf_{t\to 0} \int \int [\frac{1}{t} \int_{u_j}^{u_j + ct} f_{u_i|\boldsymbol{u}_{-i},c}(u_i) I((\boldsymbol{u}, c) \in S_1)\, du_i] f_{\boldsymbol{u}_{-i},c}(\boldsymbol{u}_{-i}, c)\, dc d\boldsymbol{u}_{-i} && (82) \\ &\ge \iint \liminf_{t\to 0} [\frac{1}{t} \int_0^{ct} f_{u_i|\boldsymbol{u}_{-i},c}(u_j + x) I((\boldsymbol{u}, c) \in S_1)\, dx] f_{\boldsymbol{u}_{-i},c}(\boldsymbol{u}_{-i}, c)\, dc d\boldsymbol{u}_{-i} && (83) \\ &= \iint c f_{u_i|\boldsymbol{u}_{-i},c}(u_j) I((\boldsymbol{u}, c) \in S_1) f_{\boldsymbol{u}_{-i},c}(\boldsymbol{u}_{-i}, c)\, d\boldsymbol{u}_{-i}\, dc \\ &= \int c [\int f_{\boldsymbol{u}|c}(u_j, \boldsymbol{u}_{-i}) I((\boldsymbol{u}, c) \in S_1)\, d\boldsymbol{u}_{-i}] f_c(c)\, dc \end{aligned}$$

where inequality (83) follows from the Fatou's Lemma. Note that the integral inside $[\cdot]$ can be infinitely large because $\int f_{\boldsymbol{u}|c}(u_j, \boldsymbol{u}_{-i})\, d\boldsymbol{u}_{-i} = f_{u_i|c}(u_j)$ can be infinitely large when $u_i = u_j$; while we can always properly select the parameters such that the constraint $I((\boldsymbol{u}, c) \in S_1)$ is satisfied by $\boldsymbol{u}_{-i}$s in a positive-measured set. Consequently, the partial derivative $\frac{\partial p_j(\boldsymbol{\tau})}{\partial \tau_i}$ can be infinitely large (i.e., not exist) at points in set $\mathcal{K}^J$, and can be unbounded near those points. That means, the arrival rate function $\boldsymbol{\Lambda}(\cdot)$ does not have to be (even locally) Lipschitz continuous.

Appendix B: Proof of Lemma 1

Given $\boldsymbol{\tau}(t-) \in \mathbb{R}_+^J$, define the following partition over the domain of $(\boldsymbol{u}, c)$ (i.e., $\mathbb{R}_+^{J+1}$):

$$\begin{aligned} \pi_0(\boldsymbol{\tau}(t-)) &:= \{(\boldsymbol{u}, c) \in \mathbb{R}_+^{J+1} \,|\, 0 > u_k - c\tau_k(t-) \text{ for all } k = 1, \ldots, J\}, \\ \pi_j(\boldsymbol{\tau}(t-)) &:= \{(\boldsymbol{u}, c) \in \mathbb{R}_+^{J+1} \,|\, u_j - c\tau_j(t-) > \max\{0, u_k - c\tau_k(t-),\ k \neq j\}\}. \end{aligned} \tag{84}$$

According to the above definition, a customer, by observing waiting-time estimates $\boldsymbol{\tau}(t-)$, will join queue $j(=0, 1, \ldots, J)$ if his parameter vector $(\boldsymbol{u}, c) \in \pi_j(\boldsymbol{\tau}(t-))$. Since a tie happens with probability zero, the probability for a customer to join queue j is given by

$$p_j(\boldsymbol{\tau}(t)) = \int_{(\boldsymbol{u},c)\in\mathbb{R}_+^{J+1}} 1((\boldsymbol{u}, c) \in \pi_j(\boldsymbol{\tau}(t-)))f(\boldsymbol{u}, c)\, d\boldsymbol{u}dc. \tag{85}$$

Let $A_j(t)$ denote the cumulative number of arrivals at queue j by time t. Let $(\boldsymbol{u}^k, c^k)$ denote the parameters of the We have

$$A_j(t) = \int_0^t 1\{(\boldsymbol{u}^{N(s)}, c^{N(s)}) \in \pi_j(\boldsymbol{\tau}(s-))\}\, dN(s), \tag{86}$$

where $N(\cdot)$ denotes a standard rate-one Poisson process. Thus, $(\boldsymbol{u}^{N(s)}, c^{N(s)})$ denote the parameters of the customer who arrive at time s. Let $\hat{A}_j(t) := \int_0^t p_j(\boldsymbol{\tau}(s-))\, ds$ denote the mean of $A(t)$. Let $\mathcal{H}$ denote the σ-field of the common probabilistic space where all the random events are defined. We then define a filtration for the arrival process as

$$\mathcal{F}(t) := \sigma(N(s),\ 0 \le s \le t) \vee \sigma((\boldsymbol{u}^{\ell\cap N(t)}, c^{\ell\cap N(t)}), \ell = 0, 1, \ldots) \vee \sigma(\mathcal{N}^0). \tag{87}$$

where $\sigma(\cdot)$ denotes the sigma-field generated by the random variables inside $(\cdot)$, and $\mathcal{N}^0$ consists of all null sets in $\mathcal{H}$. We define stochastic processes $\boldsymbol{M}^1 := (M_j^1)$ and $\boldsymbol{M}^2 := (M_j^2)$ as follows,

$$\begin{aligned} M_j^1(t) &:= A_j(t) - \textstyle\int_0^t p_j(\boldsymbol{\tau}(s-))\, dN(s) \\ &= \textstyle\int_0^t (1\{(\boldsymbol{u}^{N(s)}, c^{N(s)}) \in \pi_j(\boldsymbol{\tau}(s-)\} - p_j(\boldsymbol{\tau}(s-)))\, dN(s), \\ M_j^2(t) &:= \textstyle\int_0^t p_j(\boldsymbol{\tau}(s-))\, dN(s) - \hat{A}_j(t). \end{aligned} \tag{88}$$

We next show that M_j^1 and M_j^2 are both $\mathcal{F}(t)$-martingales. For any $t > t_0 \ge 0$, the following identify holds due to Equation (85),

$$\begin{aligned} &\mathbb{E}[M_j^1(t)|\mathcal{F}(t_0)] \\ &= M_j^1(t_0) + \mathbb{E}\left[\textstyle\int_{t_0}^t [1\{(\boldsymbol{u}^{N(s)}, c^{N(s)}) \in \pi_j(\boldsymbol{\tau}(s-))\} - p_j(\boldsymbol{\tau}(s-))]\, dN(s) \,|\, \mathcal{F}(t_0)\right] \\ &= M_j^1(t_0) + \textstyle\sum_{\ell=1}^\infty \mathbb{E}\left[\left(1\{(\boldsymbol{u}^{N(t_0)+\ell}, c^{N(t_0)+\ell}) \in \pi_j(\boldsymbol{\tau}(t_\ell-))\} - p_j(\boldsymbol{\tau}(t_\ell-))\right) 1\{t_\ell \le t\} \,|\, \mathcal{F}(t_0)\right] \\ &= M_j^1(t_0). \end{aligned} \tag{89}$$

where $t_\ell := N^{-1}(N(t_0) + \ell)$ denotes the arrival time of the $(N(t_0) + \ell)^{th}$ customer. The last equality follows that the random variables $(\boldsymbol{u}^{N(t_0)+\ell}, c^{N(t_0)+\ell})$ $(\ell = 1, 2, \ldots)$ are independent of $\mathcal{F}(t_0)$, t_ℓ, and $\boldsymbol{\tau}(t_\ell-)$. Thus, $\boldsymbol{M}^1$ is an $\mathcal{F}(t)$-martingale.

For $\boldsymbol{M^2}$, since $N(t)$ is a Poisson process, $N(t) - t$ is an $\mathcal{F}(t)$-martingale. Moreover, since $p_j(\boldsymbol{\tau}(t-))$ is left-continuous, and thus is an $\mathcal{F}(t)$-predictable process with respect to $\mathcal{F}(t)$. We then invoke the integration theorem part (β) (T8 Page 27, Brémaud (1981)), in which $X_s = p_j(\boldsymbol{\tau}(s-))$, $\lambda_u \equiv 1$, and $M_s = N(s) - s$ in the theorem. It then implies that $M_j^2(t) := \int_0^t p_j(\boldsymbol{\tau}(s-))\, dN(s) - \int_0^t p_j(\boldsymbol{\tau}(s-))\, ds$ is an $\mathcal{F}(t)$-martingale for each $j = 1, \ldots, J$. Therefore, both $\boldsymbol{M}^1$ and $\boldsymbol{M}^2$ are vector-valued $\mathcal{F}(t)$-martingale, and so is $\boldsymbol{A} - \hat{\boldsymbol{A}} = \boldsymbol{M}^1 + \boldsymbol{M}^2$.

Since $\boldsymbol{A} - \hat{\boldsymbol{A}}$ is an $\boldsymbol{F}(t)$-martingale, it must be also an $\boldsymbol{F}(t)$-local martingale. Furthermore, $\hat{\boldsymbol{A}}(0) = \mathbf{0}$. Thus, $\hat{\boldsymbol{A}}$ satisfies the definition as being a compensator of the counting process $\boldsymbol{A}(\cdot)$, i.e., the unique right-continuous and increasing process with $\hat{\boldsymbol{A}}(0) = \mathbf{0}$ such that $\boldsymbol{A} - \hat{\boldsymbol{A}}$ is a local martingale (Lowther, 2011). Furthermore, $\hat{\boldsymbol{A}}$ is a continuous compensator of $\boldsymbol{A}$ because for each $j = 1, 2, \ldots, J$, $\hat{A}_j(t) = \int_0^t p_j(\boldsymbol{\tau}(s-))\, ds$ has continuous paths (Brown and Nair, 1988). We also know that with probability 1, $\boldsymbol{A}(\cdot)$ does not have simultaneous jumps. We can then invoke Meyer's theorem (Brown and Nair, 1988) and deduce that $A_j(\hat{A}_j^{-1}(t))$, $j = 1, \ldots, J$ are independent rate-one Poisson processes, i.e.,

$$A_j(\hat{A}_j^{-1}(\cdot)) \overset{d}{=} N_j(\cdot), \tag{90}$$

where each $N_j(\cdot)$ $(j = 1, 2, \ldots, J)$ is an independent rate-one standard Poisson process. Note that the inverse function $\hat{A}_j^{-1}(\cdot)$ is well defined since $\hat{A}_j(\cdot)$ is strictly and continuously increasing. Consequently, for $0 < t_1 < \ldots \leq t_m$, we define $z_k = \hat{A}_j(t_k) = \int_0^{t_k} p_j(\boldsymbol{\tau}(s-))\, ds$ for $k = 1, 2, \ldots, m$. Then for all Borel sets $B_1, B_2, \ldots, B_m$, we have

$$\begin{aligned}
&\Pr(A_j(t_1) \in B_1, A_j(t_2) \in B_2, \ldots, A_j(t_m) \in B_m) \\
&= \Pr(A_j(\hat{A}_j^{-1}(z_1)) \in B_1, A_j(\hat{A}_j^{-1}(z_2)) \in B_2, \ldots, A_j(\hat{A}_j^{-1}(z_m)) \in B_m) \\
&= \Pr(N_1(z_1) \in B_1, N_2(z_2) \in B_2, \ldots, N_m(z_m) \in B_m) \\
&= \Pr(N_1(\textstyle\int_0^{t_1} p_j(\boldsymbol{\tau}(s-))\, ds) \in B_1, N_2(\int_0^{t_2} p_j(\boldsymbol{\tau}(s-))\, ds) \in B_2, \\
&\quad \ldots, N_m(\textstyle\int_0^{t_m} p_j(\boldsymbol{\tau}(s-))\, ds) \in B_m)
\end{aligned} \tag{91}$$

where the second equality follows from (90) (finite dimensional distribution equivalence). The above equality therefore proves the equivalence between $A_j(\cdot)$ and $N_j(\int_0^{\cdot} p_j(\boldsymbol{\tau}(s-))\, ds)$ with respect to finite dimensional distribution. Finally, since the set of discontinuous points of $\boldsymbol{\tau}(t)$ has a measure of zero, we have $N_j(\int_0^{\cdot} p_j(\boldsymbol{\tau}(s-))\, ds) = N_j(\int_0^{\cdot} p_j(\boldsymbol{\tau}(s))\, ds)$.

Appendix C: Proof of Proposition 2

Proof. If $\boldsymbol{\tau}^*$ is an equilibrium, then the arrival and departure rates must be balanced with each other in each queue. So the departure rate in each queue must be $p_j(\boldsymbol{\tau}^*)$. For queues with excessive service capacity, we must have $\mu_j - p_j(\boldsymbol{\tau}^*) > 0$, and that queue must be empty so $\tau_j^* = 0$; for other queues, we have $\mu_j - p_j(\boldsymbol{\tau}) = 0$. We thus proved the complementary slackness condition in (40). The other inequality constraints can be proved straightforwardly.

Suppose $\boldsymbol{\tau}^*$ is a solution to (40). For queues with $\tau_j^* > 0$, by the complementary slackness condition in (40), we have $\mu_j - p_j(\boldsymbol{\tau}) = 0$, which implies that the service rate and arrival rate are balanced for those queues; for queues with $\tau_j^* = 0$, we know that the arrival rate has not exceeded the service capacity due to the inequality constraint $\mu_j - p_j(\boldsymbol{\tau}) \geq 0$. Since those queues are empty, the arrival and departure rates must be balanced. Thus, the drift coefficient in equation (22) must equal to zero at $\boldsymbol{\tau}^*$, which implies $\boldsymbol{\tau}(t) \equiv \boldsymbol{\tau}^*$ provided that $\boldsymbol{\tau}(t)$ is a solution to (22) with $\boldsymbol{\tau}(0) = \boldsymbol{\tau}^*$. ■

Appendix D: Proof of Theorem 2

Proof. We first use (CD-a) and (CD-c) to prove that $-\boldsymbol{\Lambda}(\cdot) := -(p_j(\cdot))_{j=1,\ldots,J}$ satisfies the so-called P-property (Moré and Rheinboldt (1973)). Then by Theorem 2.3 of Moré (1974a) or the comments after Theorem 1.6 of Megiddo and Kojima (1977), the P-property of $-\boldsymbol{\Lambda}(\boldsymbol{\tau})$ ensures that the solution to the NCP (40) is unique, if exists.

$$\text{P-Property: } \forall \boldsymbol{\tau}^1, \boldsymbol{\tau}^2 \in \mathbb{R}_+^J,\ \boldsymbol{\tau}^1 \neq \boldsymbol{\tau}^2,\ \min_{j=1}^{J}\ (\tau_j^1 - \tau_j^2)(p_j(\boldsymbol{\tau}^1) - p_j(\boldsymbol{\tau}^2)) < 0. \tag{92}$$

Without loss of generality, we assume that $\tau_{j^*}^1 - \tau_{j^*}^2 = \max_j(\tau_j^1 - \tau_j^2) > 0$ for some j^*, and define

$$\overline{\Delta}\tau := \tau_{j^*}^1 - \tau_{j^*}^2. \tag{93}$$

Then to prove (92), it suffices to prove that $p_{j^*}(\boldsymbol{\tau}^1) < p_{j^*}(\boldsymbol{\tau}^2)$. By the definition of $\overline{\Delta}\tau$, we have $\boldsymbol{\tau}^1 \le \boldsymbol{\tau}^2 + \overline{\Delta}\tau e$, but $\boldsymbol{\tau}^1_{j^*} = \boldsymbol{\tau}^2_{j^*} + \overline{\Delta}\tau$. Therefore, (CD-a) implies that

$$p_{j^*}(\boldsymbol{\tau}^1) \le p_{j^*}(\boldsymbol{\tau}^2 + \overline{\Delta}\tau e). \tag{94}$$

If we define a univariate function $f(x) := p_{j^*}(\boldsymbol{\tau}^2 + x\boldsymbol{e})$ and apply the mean value theorem to $f(\cdot)$[3], we get

$$f(\overline{\Delta}\tau) - f(0) = \overline{\Delta}\tau f'(\zeta). \tag{95}$$

for some $\zeta \in [0, \overline{\Delta}\tau]$. That implies

$$\begin{aligned} p_{j^*}(\boldsymbol{\tau}^2 + \overline{\Delta}\tau \boldsymbol{e}) - p_{j^*}(\boldsymbol{\tau}^2) &= \overline{\Delta}\tau \textstyle\sum_i R_{j^*i}(\boldsymbol{\tau}^2 + \zeta\boldsymbol{e}) \\ &= \overline{\Delta}\tau R_{j^*j^*}(\boldsymbol{\tau}^2 + \zeta\boldsymbol{e}) + \overline{\Delta}\tau \textstyle\sum_{i\neq j^*} R_{j^*i}(\boldsymbol{\tau}^2 + \zeta\boldsymbol{e}) \\ &< 0 \end{aligned} \tag{96}$$

for some $\zeta \in [0, \overline{\Delta}\tau]$, where $R_{ji}(\boldsymbol{\tau}^2 + \zeta\boldsymbol{e})$ represents the entry at the j^{th} row and i^{th} column of the Jacobean matrix evaluated at $\boldsymbol{\tau}^2 + \zeta\boldsymbol{e}$, and the last inequality follows from (CD-c). Inequalities (94) and (96) together imply that $p_{j^*}(\boldsymbol{\tau}^1) < p_{j^*}(\boldsymbol{\tau}^2)$, which leads to the P-property.

We next prove the existence of a solution to the NCP. The most well known sufficient conditions for existence is that the Jacobian of $-\boldsymbol{\Lambda}(\boldsymbol{\tau})$ is positively bounded, i.e., every principle minor of the Jacobian of $-\boldsymbol{\Lambda}(\boldsymbol{\tau})$ is bounded between $[\delta, \delta^{-1}]$ for all $\boldsymbol{\tau}$ (Cottle (1966)), or that $-\boldsymbol{\Lambda}(\boldsymbol{\tau})$ is a uniform P-function, i.e., $\min(\tau^1_j - \tau^2_j)(p_j(\boldsymbol{\tau}^1) - p_j(\boldsymbol{\tau}^2)) \le -c\|\boldsymbol{\tau}^1 - \boldsymbol{\tau}^2\|^2$ for some $c > 0$ (Karamardian (1969); Moré (1974b)). Unfortunately, neither condition is satisfied by our $-\boldsymbol{\Lambda}(\boldsymbol{\tau})$, as its Jacobian can be arbitrarily close to a singular matrix when $\|\boldsymbol{\tau}\| \to \infty$.

The next step of the proof involves proposing a new set of sufficient conditions for the existence of a solution to an NCP of the form of (40), i.e., (CD-a), (CD-b), and the stability condition (9). Note that (CD-c) is only needed to prove the uniqueness of the solution, but not the existence.

We use a constructive approach to prove the existence of the equilibrium. We prove that the equilibrium state can be achieved by iterative adjustment of the waiting times $\boldsymbol{\tau}$. This adjustment process is referred to as a tatonnement process in the economics literature Arrow et al. (1959); Walras (2013). We start with $\boldsymbol{\tau} = \boldsymbol{0}$. In each iteration, we check sequentially if $\mu_j - p_j(\boldsymbol{\tau}) < 0$ for each $j = 1, 2, \dots, J$. Suppose for some j, $\mu_j - p_j(\boldsymbol{\tau}) < 0$, then we increase the value of τ_j and keep the other components of $\boldsymbol{\tau}$ unchanged until $\mu_j - p_j(\boldsymbol{\tau}) = 0$. Such a $\boldsymbol{\tau}$ always exists because $\liminf \mu_j - p_j(\boldsymbol{\tau}) > 0$ by the stability condition (9), and $\mu_j - p_j(\boldsymbol{\tau})$ increases continuously in τ_j by (CD-b). We repeat the above procedure sequentially for $j = 1, 2, \dots, J$ until at some j, $\mu_k - p_k(\boldsymbol{\tau}) \ge 0$ for $k > j$. Note that after τ_j being increased, the value of $\mu_l - p_l(\boldsymbol{\tau})$ can only decrease and turn negative again for some $\ell < j$ due to (CD-a). Therefore, we have to run the above algorithm for another iteration, that is, checking if $\mu_j - p_j(\boldsymbol{\tau}) < 0$ for some j and increase τ_j to make the equality to hold.

According to the above discussion, either at the very beginning $\mu_j - p_j(\boldsymbol{\tau}) > 0$, or $\mu_j - p_j(\boldsymbol{\tau}) \le 0$ throughout the entire algorithm. We use $\boldsymbol{\tau}^N$ to denote the updated value of $\boldsymbol{\tau}$ in the N^{th} iteration. If in some iteration N, $\mu_j - p_j(\boldsymbol{\tau}^N) \ge 0$ for all j, then $\boldsymbol{\tau}^N$ is a solution to the NCP because $\mu_j - p_j(\boldsymbol{\tau}) > 0$ only if the value of τ_j has never been updated (so $\tau_j = 0$); otherwise, we obtain a sequence of waiting-time vectors $\{\boldsymbol{\tau}^N | N = 1, 2, \dots\}$. We next show that $\boldsymbol{\tau}^N \to \boldsymbol{\tau}^* < \infty$ and $\boldsymbol{\tau}^*$ is the unique solution to the NCP (40).

Without loss of generality, we assume that the value of τ_j has been updated (so $\tau_j > 0$) at iteration $N_1, N_2, \dots, N_l, \dots$. After each time τ_j was updated, the waiting-time vector $\boldsymbol{\tau} =$

[3] The mean value theorem holds even if at some point x, the derivative $f'(x)$ may equal to $+\infty$ or $-\infty$, as long as $f'(x)$ has no jumps.

$(\tau_1^{N_l},\dots,\tau_j^{N_l},\tau_{j+1}^{N_l-1},\dots,\tau_J^{N_l-1})$ must solve the equation $\mu_j - p_j(\boldsymbol{\tau}) = 0$. Therefore, the following equation must hold for each $l = 1,2,\dots$,

$$\mu_j - p_j(\tau_1^{N_l},\dots,\tau_j^{N_l},\tau_{j+1}^{N_l-1},\dots,\tau_J^{N_l-1}) = 0. \tag{97}$$

Since the value of τ_j^N can only increase after each iteration, the monotone convergence theorem implies that $\tau_j \to \tau_j^*$. By the stability condition (9), τ_j^* must be a finite number, otherwise we have $\mu_j - p_j(\boldsymbol{\tau}^N) \to \mu_j - 0 > 0$, which contradicts the complementarity slackness condition. By letting $l \to \infty$ and taking the limit on both sides of equation (97), we get $\mu_j - p_j(\boldsymbol{\tau}^*) = 0$. By repeatedly applying this argument for $j = 1,2,\dots,J$, we prove that $(\boldsymbol{\mu} - \boldsymbol{\Lambda}(\boldsymbol{\tau}^*),\boldsymbol{\tau}^*)$ is a solution to the NCP (40). ■

Appendix E: Proof of Theorem 3

Proof. We define $\overline{\Delta}\boldsymbol{\tau}(t) = \max_j \tau_j(t) - \tau_j^*(t)$ and $\underline{\Delta}\boldsymbol{\tau}(t) = \min_j \tau_j(t) - \tau_j^*(t)$. We first prove that for any $\delta > 0$, if $\overline{\Delta}\boldsymbol{\tau}(t) > \delta$, then $\overline{\Delta}\boldsymbol{\tau}'(t) \le -h(\delta)$, where $h(\delta)$ is a positive constant which depends on the value of δ.

Suppose $\tau_{j^*}(t) - \tau_{j^*}^* = \overline{\Delta}\boldsymbol{\tau}(t) \ge \delta$. Since $\tau_{j^*}(t) > 0$, the complementarity slackness condition implies that $\mu_j = p_j(\boldsymbol{\tau}^*)$. Thus,

$$\tau_{j^*}'(t) = \frac{X_{j^*}'(t)}{\mu_{j^*}} = \frac{p_{j^*}(\boldsymbol{\tau}(t))}{\mu_{j^*}} - 1 = \frac{p_{j^*}(\boldsymbol{\tau}(t))}{p_{j^*}(\boldsymbol{\tau}^*)} - 1. \tag{98}$$

Note that $\tau_{j^*}'(t)$ exists a.e., because $X_{j^*}(t)$ can be expressed as integrals from 0 to t (See e.g., Equation (31)) and is therefore absolute continuous.

With the above equality, to show that $\tau_{j^*}'(t) \le -h(\delta)$, it suffices to show that

$$\frac{p_{j^*}(\boldsymbol{\tau}(t)) - p_{j^*}(\boldsymbol{\tau}^*)}{p_{j^*}(\boldsymbol{\tau}^*)} \le -h(\delta). \tag{99}$$

We prove the above inequality using a similar argument as in the proof of P-property of Theorem 2. By substituting $\boldsymbol{\tau}^1 = \boldsymbol{\tau}(t)$ and $\boldsymbol{\tau}^2 = \boldsymbol{\tau}^*$ into inequality (94) and (96), we get

$$\begin{aligned} p_{j^*}(\boldsymbol{\tau}(t)) - p_{j^*}(\boldsymbol{\tau}^*) &\le p_{j^*}(\boldsymbol{\tau}^* + \overline{\Delta}\boldsymbol{\tau}(t)e) - p_{j^*}(\boldsymbol{\tau}^*) \\ &\le p_{j^*}(\boldsymbol{\tau}^* + \delta e) - p_{j^*}(\boldsymbol{\tau}^*) \\ &= \delta R_{j^*j^*}(\boldsymbol{\tau}^* + \zeta e) + \delta \textstyle\sum_{i\neq j^*} R_{j^*i}(\boldsymbol{\tau}^* + \zeta e) \end{aligned} \tag{100}$$

for some $\zeta \in [0,\delta]$. In Equation (100), the first inequality follows from inequality (94) (which uses property (CD-a)), and the second inequality follows from $\overline{\Delta}\boldsymbol{\tau}(t) \ge \delta$ and property (CD-c). We then define

$$h(\delta) := \frac{-\delta}{p_j(\boldsymbol{\tau}^*)}\left(\max\{z \in [0,\delta] \mid R_{j^*j^*}(\boldsymbol{\tau}^* + ze) + \sum_{i\neq j^*} R_{j^*i}(\boldsymbol{\tau}^* + ze)\}\right). \tag{101}$$

Using (CD-c), we deduce that $R_{j^*j^*}(\boldsymbol{\tau}^* + ze) + \sum_{i\neq j^*} R_{j^*i}(\boldsymbol{\tau}^* + ze) < 0$ for all $z \in [0,\delta]$. Therefore, $h(\delta)$ is a positive constant that is independent of $\boldsymbol{\tau}(t)$. With $h(\delta)$ defined as in (101), inequality (100) directly implies (99). Therefore, $\tau_{j^*}'(t) = \overline{\Delta}\boldsymbol{\tau}'(t) \le -h(\delta)$ whenever $\overline{\Delta}\boldsymbol{\tau}(t) \ge \delta$. An analogous argument can be used to prove that $\underline{\Delta}\boldsymbol{\tau}'(t) \ge h(\delta)$ whenever $\underline{\Delta}\boldsymbol{\tau}(t) \le -\delta$. Therefore, whenever the maximum deviation of $\boldsymbol{\tau}(t)$ from $\boldsymbol{\tau}^*$ has to decrease at a rate of at least $h(\delta)$ whenever it is greater than δ. This guarantees that the maximum deviation must drop below δ after a finite period. The conclusion of Theorem 3 then follows by letting $\delta \to 0$. ■

Appendix F: Proof of Lemma 3

We define $n^{1/2}\Delta\boldsymbol{\tau}^n(s)$ for a given $\boldsymbol{Q}^n(s)$ as

$$\begin{aligned} n^{1/2}\Delta\boldsymbol{\tau}^n(s) &= n^{1/2}(n^{1/2}\boldsymbol{Q}^n(s)+n\boldsymbol{\tau}^*\circ\boldsymbol{\mu})\circ(n\boldsymbol{\mu}^n)^{-1}-\boldsymbol{\tau}^{n,*} \\ &= \boldsymbol{Q}^n(s)\circ(\boldsymbol{\mu}^n)^{-1}+(\boldsymbol{\tau}^*\circ\boldsymbol{\mu}-\boldsymbol{\tau}^{n,*}\circ\boldsymbol{\mu}^n)\circ(\boldsymbol{\mu}^n)^{-1} \end{aligned} \tag{102}$$

Note that the second term at the RHS of (102) converges to $-\boldsymbol{\vartheta}\circ\boldsymbol{\mu}^{-1}$, so the second term must be bounded for all n. Also, the sequence $\{\boldsymbol{\mu}^n\}$ is bounded as it converges to $\boldsymbol{\mu}$. Thus, there exists $\epsilon>0$, such that for sufficiently large n,

$$n^{1/2}\Delta\boldsymbol{\tau}_j^n(s)+\frac{\vartheta_j}{\mu_j}-\epsilon\le\frac{\boldsymbol{Q}_j^n(s)}{\mu_j}\le n^{1/2}\Delta\boldsymbol{\tau}_j^n(s)+\frac{\vartheta_j}{\mu_j}+\epsilon, \tag{103}$$

which implies that $n^{1/2}\Delta\boldsymbol{\tau}^n(s)$ is bounded if and only if $\boldsymbol{Q}^n(s)$ is bounded. We let $\overline{\Delta}\boldsymbol{\tau}^n(t)$ and $\underline{\Delta}\boldsymbol{\tau}^n(t)$ denote the maximal and minimal entries in the vector $\Delta\boldsymbol{\tau}^n(t)$, respectively. To prove Lemma 3, it suffices to prove that for any fixed $T>0$, when $\kappa\to\infty$,

$$\begin{aligned} &\limsup\nolimits_n \Pr(\sup\{n^{1/2}\overline{\Delta}\boldsymbol{\tau}^n(t)\mid t\in[0,T]\}>\kappa) &\to 0 \\ &\limsup\nolimits_n \Pr(\inf\{n^{1/2}\underline{\Delta}\boldsymbol{\tau}^n(t)\mid t\in[0,T]\}<-\kappa) &\to 0 \end{aligned} \tag{104}$$

To prove (104), we first derive an expression for $\boldsymbol{Q}^n$ in analogue to the expression for $\boldsymbol{Q}^{\kappa,n}$ in (50) by ignoring the reflection barrier at $\pm\kappa$,

$$Q_j^n(t)=Q_j^n(0)+\textstyle\int_0^t\Gamma_j^n(\boldsymbol{\tau}^{n,*}+\Delta\boldsymbol{\tau}^n(s))ds+n^{-1/2}Z_j^n(t)+n^{-1/2}L_j^n(t), \tag{105}$$

where $\Delta\boldsymbol{\tau}^n(s)$ is defined as in (102) for a given $\boldsymbol{Q}^n(s)$, $\Gamma_j^n(\boldsymbol{\tau}):=n^{1/2}\left(p_j(\boldsymbol{\tau})-\mu_j^n\right)$ represents the deterministic drift that can be non-Lipschitz, and $Z_j^n(t)$ represents a mean-zero stochastic process which was defined in Equation (32).

We next consider the scenario when $n^{1/2}\overline{\Delta}\boldsymbol{\tau}^n(s)=n^{1/2}(\tau_{j^*}^n(s)-\tau_{j^*}^{n,*})>\delta$ in some interval $[a_1,b_1]$ and for some fixed $j^*\in\{j=1,\dots,J\}$. That means, $\boldsymbol{\tau}^n$ has the largest positive deviation from the equilibrium $\boldsymbol{\tau}^{n,*}$ along dimension j^* over $[a_1,b_1]$. Then using the choice-driven property of $\boldsymbol{\Gamma}^n(\boldsymbol{\tau})$ (whose Jacobean is $\boldsymbol{R}(\boldsymbol{\tau})$ so it inherits the choice-driven property), we can prove that over $[a_1,b_1]$, the drift term would be upper bounded by a negative constant (See (110) below), and consequently the deviation $\overline{\Delta}\boldsymbol{\tau}^n(s)$ would decrease by at least an amount proportional to b_1-a_1 (See (113)).

Formally, we have

$$\begin{aligned} \Gamma_{j^*}^n(\boldsymbol{\tau}^{n,*}+\Delta\boldsymbol{\tau}^n(s)) &= n^{1/2}\left(p_{j^*}(\boldsymbol{\tau}^{n,*}+\Delta\boldsymbol{\tau}^n(s))-\mu_{j^*}^n\right) \\ &= n^{1/2}\left(p_{j^*}(\boldsymbol{\tau}^{n,*}+\Delta\boldsymbol{\tau}^n(s))-p_{j^*}(\boldsymbol{\tau}^{n,*})\right)+n^{1/2}(p_{j^*}(\boldsymbol{\tau}^{n,*})-\mu_{j^*}^n). \end{aligned} \tag{106}$$

We next provide an upper bound for the RHS of Equation (106). In inequality (99) (which builds on the choice-driven property), by replacing $\boldsymbol{\tau}(t)$ with $\boldsymbol{\tau}^{n,*}+\Delta\boldsymbol{\tau}^n(s)$, and by noting that $\overline{\Delta}\tau^n(s)\ge n^{-1/2}\delta$, we get

$$p_{j^*}(\boldsymbol{\tau}^{n,*}+\Delta\tau^n(s))-p_{j^*}(\boldsymbol{\tau}^{n,*})\le -n^{-1/2}h^n(\delta). \tag{107}$$

where $h^n(\cdot)$ follows a similar functional form of $h(\cdot)$ as given in Equation (101), that is,

$$h^n(\delta):=\frac{-\delta}{p_j(\boldsymbol{\tau}^{n,*})}\left(\max\{z\in[0,n^{-1/2}\delta]\mid R_{j^*j^*}(\boldsymbol{\tau}^{n,*}+ze)+\sum_{i\neq j^*}R_{j^*i}(\boldsymbol{\tau}^{n,*}+ze)\}\right)(>0) \tag{108}$$

Inequality (107) allows us to upper bound the RHS of (106) as

$$\begin{aligned} \Gamma_{j^*}^n(\boldsymbol{\tau}^{n,*}+\Delta\boldsymbol{\tau}^n(s)) &\le -h^n(\delta)+n^{1/2}(p_{j^*}(\boldsymbol{\tau}^{n,*})-\mu_{j^*}^n) \\ &\to \tfrac{\delta}{p_j(\boldsymbol{\tau}^{n,*})}\left(R_{j^*j^*}(\boldsymbol{\tau}^{n,*})+\textstyle\sum_{i\neq j^*}R_{j^*i}(\boldsymbol{\tau}^{n,*})\right)-\theta_{j^*} \end{aligned} \tag{109}$$

That means, for sufficiently large n,

$$\Gamma^n_{j^*}(\boldsymbol{\tau}^{n,*}+\Delta\boldsymbol{\tau}^n(s)) < \frac{\delta}{p_j(\boldsymbol{\tau}^{n,*})}\left(R_{j^*j^*}(\boldsymbol{\tau}^{n,*})+\sum_{i\neq j^*}R_{j^*i}(\boldsymbol{\tau}^{n,*})\right)-\theta_{j^*} := -\Delta_n < 0 \tag{110}$$

where $R_{j^*j^*}(\boldsymbol{\tau}^{n,*})+\sum_{i\neq j^*}R_{j^*i}(\boldsymbol{\tau}^{n,*})<0$ by the choice-driven property. By looking into the sequence $\{\Delta_n\}$, we deduce that it converges to some positive constant, $\Delta>0$. Inequalities (105) and (110) imply that

$$Q^n_{j^*}(b_1)-Q^n_{j^*}(a_1)\le -\Delta_n(b_1-a_1)+n^{-1/2}(Z^n_{j^*}(b_1)-Z^n_{j^*}(a_1))+n^{-1/2}(L_{j^*}(b_1)-L_{j^*}(a_1)) \tag{111}$$

If $j^*\in\mathcal{J}^-\cup\mathcal{J}^+$, then $\tau^n_{j^*}(s)-\tau^{n,*}_{j^*}>0$ implies that $Q^n_{j^*}(s)>0$ over $[a_1,b_1)$. Consequently, $L_{j^*}(b_1)-L_{j^*}(a_1)=0$. If $j^*\in\mathcal{J}^{++}$, then there is no reflection barrier along dimension j^*, so $L_{j^*}\equiv 0$. Thus in either case, $L_i(b_1)-L_i(a_1)=0$ and inequality (111) implies that

$$Q^n_{j^*}(b_1)-Q^n_{j^*}(a_1)\le -\Delta_n(b_1-a_1)+n^{-1/2}(Z^n_i(b_1)-Z^n_i(a_1)). \tag{112}$$

which leads to

$$\begin{aligned} n^{1/2}(\overline{\Delta}\tau^n(b_1)-\overline{\Delta}\tau^n(a_1)) &= n^{1/2}(\tau^n_{j^*}(b_1)-\tau^n_{j^*}(a_1))\\ &= \tfrac{1}{\mu^n_j}\left(Q^n_{j^*}(b_1)-Q^n_{j^*}(a_1)\right).\\ &\le \tfrac{1}{\mu^n_j}\left(-\Delta_n(b_1-a_1)+n^{-1/2}(Z^n_{j^*}(b_1)-Z^n_{j^*}(a_1))\right)\end{aligned} \tag{113}$$

That means, the largest deviation $\overline{\Delta}\tau^n$ keeps decreasing. For any interval $[a,b]\subseteq[0,T]$ over which $n^{1/2}\overline{\Delta}\tau(s)\ge\delta$, we can partition $[a,b]$ in into countably many intervals $\cup_{i=1}^{\infty}[a_i,b_i)$ such that $\overline{\Delta}\tau(s)=\tau^n_{j^i}(s)-\tau^{n,*}_{j^i}$ for the same index $j^i\in\{1,2,\dots,J\}$ and for all $s\in[a_i,b_i)$. Using this notation, we derive the following inequality

$$\begin{aligned} n^{1/2}(\overline{\Delta}\tau^n(b)-\overline{\Delta}\tau^n(a)) &= \textstyle\sum_{i=1}^{\infty} n^{1/2}(\overline{\Delta}\tau^n(b_i)-\overline{\Delta}\tau^n(a_i))\\ &\le \textstyle\sum_{i=1}^{\infty}\frac{1}{\mu^n_{j^i}}\left(-\Delta_n(b_i-a_i)+n^{-1/2}(Z^n_{j^i}(b_i)-Z^n_{j^i}(a_i))\right)\\ &\le \tfrac{1}{\min_j\mu^n_j}\left(-\Delta_n(b-a)+n^{-1/2}\|\boldsymbol{Z}^n(b-a)\|\right)\end{aligned} \tag{114}$$

Now let $\delta=\frac{\kappa}{2}$. If $\overline{\Delta}\tau^n(\cdot)$ has ever exceeded $\frac{\kappa}{2}$ over $[0,t]$, then we let $a=\sup\{s\in[0,t]:\overline{\Delta}\tau^n(s)\le\frac{\kappa}{2}\}$ and $b=t$. The selection of a and b guarantees that $\overline{\Delta}\tau^n(a)=\frac{\kappa}{2}$ and $\overline{\Delta}\tau^n(s)\ge\frac{\kappa}{2}$ for all $s\in[a,b]$. Thus, Equation (114) implies that[4]

$$\begin{aligned} n^{1/2}\overline{\Delta}\tau^n(t)-\tfrac{\kappa}{2} &= n^{1/2}(\overline{\Delta}\tau^n(b)-\overline{\Delta}\tau^n(a))\\ &\le \tfrac{1}{\min_j\mu^n_j}\left(n^{-1/2}\|\boldsymbol{Z}^n\|_t\right).\end{aligned} \tag{115}$$

If $\overline{\Delta}\tau(\cdot)$ is always upper bounded by $\frac{\kappa}{2}$ over $[0,t]$, then the above inequality holds trivially. We thus have

$$\begin{aligned} n^{1/2}\sup\{\overline{\Delta}\tau^n(t)\mid t\in[0,T]\} &\le \tfrac{\kappa}{2}+\tfrac{1}{\min_j\mu^n_j}n^{-1/2}\sup\{\|\boldsymbol{Z}^n(t)\|\mid t\in[0,T]\}\\ &= \tfrac{\kappa}{2}+\tfrac{1}{\min_j\mu^n_j}n^{-1/2}\|\boldsymbol{Z}^n\|_T.\end{aligned} \tag{116}$$

[4] To derive (115), we have only used a weaker upper bound (114) for $\overline{\Delta}\tau^n(b)-\overline{\Delta}\tau^n(a)$ by ignoring the negative drift $-\Delta_n(b-a)$. The original upper bound (114) including $-\Delta_n(b-a)$, however, is needed in the later proof for Proposition 4.

When $\kappa \to \infty$, we deduce that

$$\begin{aligned}
&\limsup_n \Pr(\sup\{n^{1/2}\overline{\Delta}\tau^n(t) \mid t \in [0,T]\} > \kappa) \\
\leq &\limsup_n \Pr(\sup\{n^{1/2}\overline{\Delta}\tau^n(t) \mid t \in [0,T]\} > \kappa \mid n^{1/2}\overline{\Delta}\tau^n(0) \leq \tfrac{\kappa}{2}) \Pr(n^{1/2}\overline{\Delta}\tau^n(0) \leq \tfrac{\kappa}{2}) \\
&+ \limsup_n \Pr(n^{1/2}\overline{\Delta}\tau^n(0) > \tfrac{\kappa}{2}) \\
\to &\limsup_n \Pr(\sup\{n^{1/2}\overline{\Delta}\tau^n(t) \mid t \in [0,T]\} > \kappa \mid n^{1/2}\overline{\Delta}\tau^n(0) \leq \tfrac{\kappa}{2}) \cdot 1 + 0 \\
\leq &\limsup_n \Pr(\sup_{t\in[0,T]} \tfrac{1}{\min_j \mu_j^n} n^{-1/2}\|\boldsymbol{Z}^n\|_T > \tfrac{\kappa}{2}) \\
\leq &\sup_n 2c_1 \exp(-\tfrac{c_2}{4}\kappa^2) + 2n^{c_3}\exp(-\tfrac{c_4}{2}\kappa\sqrt{n})
\end{aligned} \tag{117}$$

for some positive constants c_i $(i = 1,2,3,4)$. In Equation (117), the convergence result follows from $\limsup_n \Pr(n^{1/2}\overline{\Delta}\tau^n(0) > \frac{\kappa}{2}) \to 0$ as $\boldsymbol{Q}^n(0)$ (so $n^{1/2}\Delta\boldsymbol{\tau}^n(0)$) is assumed to have finite expectation; the second inequality follows from (116), and the last inequality follows from the upper bound (134) for the tail probability of $n^{-1/2}\|\boldsymbol{Z}^n\|_T$ (See Lemma 4 in Appendix J). Note that the second term of RHS in Equation (117) is dominated by $\exp(-\frac{c_4}{4}\kappa\sqrt{n})$ when n is large, so the RHS has to converge to zero when $\kappa \to \infty$, which leads to the first convergence equation in (104).

The second convergence in (104) can be proved using an analogous argument and is omitted here.

Appendix G: Proof of Proposition 3

Proof. According to Example 3.10, Claim 1 of Kang and Ramanan (2014), if the diffusion limit is a solution to an SDER with affine drift coefficient $\boldsymbol{C}\boldsymbol{x}$, and if $\boldsymbol{C}^* := [\boldsymbol{A} - \overline{\boldsymbol{N}}^{-1}\overline{\boldsymbol{Q}}]^{-1}\boldsymbol{C}$ (see definitions in Kang and Ramanan (2014)) is symmetric, then $p(\boldsymbol{x}) = e^{\boldsymbol{x}^T\boldsymbol{C}_*\boldsymbol{x}}$, after normalization, gives the stationary distribution of the diffusion limit. We next check whether with the parameters in our setting, $\boldsymbol{C}_*$ is symmetric and $p(\boldsymbol{x})$ is proportional to $\pi(\boldsymbol{z})$ as defined in the proposition. Because in our model the reflection direction is always normal, it has zero component tangential to the boundary. Thus, we have $\overline{\boldsymbol{Q}} = 0$, because its rows are exactly the tangential components of the reflection direction according to the comments after Theorem 3 in Kang and Ramanan (2014). Consequently, by comparing the SDER in Kang and Ramanan (2014) to Equation (47), we have $\boldsymbol{A} = \boldsymbol{\Sigma}^2 = (1+\omega_1^2)\mathrm{Diag}\,(\boldsymbol{\mu})$, $\boldsymbol{x} = \boldsymbol{z} - \boldsymbol{\vartheta} - (\mathrm{Diag}\,(\boldsymbol{\mu})\boldsymbol{R}^*)^{-1}\boldsymbol{\theta}$ and $\boldsymbol{C} = \boldsymbol{R}^*\mathrm{Diag}\,(\boldsymbol{\mu}^{-1})$. Thus, $\boldsymbol{C}^* := \boldsymbol{A}^{-1}\boldsymbol{C} = (1+\omega_1^2)^{-1}\mathrm{Diag}\,(\boldsymbol{\mu}^{-1})\boldsymbol{R}^*\mathrm{Diag}\,(\boldsymbol{\mu}^{-1})$ is symmetric and negative definite as $\boldsymbol{R}^*$ is symmetric and negative definite. We thus conclude that

$$\begin{aligned}
&p(\boldsymbol{x}) = \exp(\boldsymbol{x}^T\boldsymbol{C}_*\boldsymbol{x}) \\
= &\exp((\boldsymbol{z} - \boldsymbol{\vartheta} - (\mathrm{Diag}\,(\boldsymbol{\mu})\boldsymbol{R}^*)^{-1}\boldsymbol{\theta})^T((1+\omega_1^2)^{-1}\mathrm{Diag}\,(\boldsymbol{\mu}^{-1})\boldsymbol{R}^*\mathrm{Diag}\,(\boldsymbol{\mu}^{-1}))(\boldsymbol{z} - \boldsymbol{\vartheta} - (\mathrm{Diag}\,(\boldsymbol{\mu})\boldsymbol{R}^*)^{-1}\boldsymbol{\theta})) \\
= &\exp(-\tfrac{1}{2}(\boldsymbol{z} - \boldsymbol{\vartheta} - (\mathrm{Diag}\,(\boldsymbol{\mu})\boldsymbol{R}^*)^{-1}\boldsymbol{\theta})^T(-\tfrac{1}{2}(1+\omega_1^2)\mathrm{Diag}\,(\boldsymbol{\mu})(\boldsymbol{R}^*)^{-1}\mathrm{Diag}\,(\boldsymbol{\mu}))^{-1} \\
&\qquad(\boldsymbol{z} - \boldsymbol{\vartheta} - (\mathrm{Diag}\,(\boldsymbol{\mu})\boldsymbol{R}^*)^{-1}\boldsymbol{\theta}))
\end{aligned} \tag{118}$$

is proportional to the density of the stationary distribution of the diffusion limit, $\pi_{\boldsymbol{Y}}(\boldsymbol{z})$. By looking into the above expression, we find that $p(\boldsymbol{x})$ is proportional to the density of a multivariate Gaussian random variable with mean $\boldsymbol{\vartheta} + (\mathrm{Diag}\,(\boldsymbol{\mu})\boldsymbol{R}^*)^{-1}\boldsymbol{\theta}$ and covariance matrix $-\frac{1}{2}(1+\omega_1^2)\mathrm{Diag}\,(\boldsymbol{\mu})(\boldsymbol{R}^*)^{-1}\mathrm{Diag}\,(\boldsymbol{\mu})$, which is denoted by $\pi(\boldsymbol{z})$. Therefore, $\pi_{\boldsymbol{Y}}(\boldsymbol{z})$ is proportional to $\pi(\boldsymbol{z})$. Normalizing $\pi(\boldsymbol{z})$ thus leads to an exact expression for $\pi_{\boldsymbol{Y}}(\boldsymbol{z})$ in (55). ■

Appendix H: Proof of Proposition 4

Equation (103) implies that when n is sufficiently large, the difference between $V(\Xi^n(t))(= \|\boldsymbol{Q}^n\|^{\boldsymbol{\mu}^{-1}})$ and $\|n^{1/2}\Delta\boldsymbol{\tau}^n(t)\|$ is almost a constant (i.e., within $\pm\epsilon$). So proving Equation (58) is equivalent to proving the same bounded condition for $\|n^{1/2}\Delta\boldsymbol{\tau}^n(t)\|$, that is, for some $u_0 > 0$, $t_0 \geq 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{\Xi^n(0)\in\Omega} \mathbb{E}[\exp\left(u_0(\|n^{1/2}\Delta\boldsymbol{\tau}^n(t_0)\| - \|n^{1/2}\Delta\boldsymbol{\tau}^n(0)\|)^+\right) \mid \Xi^n(0)] < \infty \\
&\limsup_{n\to\infty} \sup_{\Xi^n(0)\in\Omega} \mathbb{E}[(\|n^{1/2}\Delta\boldsymbol{\tau}^n(t_0)\| - \|n^{1/2}\Delta\boldsymbol{\tau}^n(0)\|)^2 \\
&\qquad\qquad \exp\left(u(\|n^{1/2}\Delta\boldsymbol{\tau}^n(t_0)\| - \|n^{1/2}\Delta\boldsymbol{\tau}^n(0)\|)^+\right) \mid \Xi^n(0)] < \infty
\end{aligned} \tag{119}$$

To prove (119), we first consider the case when $\|n^{1/2}\Delta\boldsymbol{\tau}^n(s)\| > \frac{\kappa}{2}$ for all $s \in [0,T]$. By Equation (114) (which builds on the choice-driven properties of the arrival rate) and by plugging into $a = 0$ and $b = t_0$, we have

$$n^{1/2}\|\Delta\tau^n(t)\| - n^{1/2}\|\Delta\tau^n(0)\| \leq \tfrac{1}{\min_j \mu_j^n}\left(-\Delta_n t_0 + n^{-1/2}\|\boldsymbol{Z}^n(t_0)\|\right) \tag{120}$$

where Δ_n was defined in (110), which converges to a positive constant $\Delta > 0$. By choosing

$$t_0 = \frac{\min_j \mu_j^n}{\Delta}\left(n^{1/2}\|\Delta\tau^n(0)\| - \frac{\kappa}{2}\right)^+, \tag{121}$$

for sufficiently large n, Equation (120) implies that

$$n^{1/2}\|\Delta\tau^n(t)\| \leq \frac{\kappa}{2} + \frac{1}{\min_j \mu_j^n} n^{-1/2}\|\boldsymbol{Z}^n(t_0)\|. \tag{122}$$

In the other case when $\|n^{1/2}\Delta\boldsymbol{\tau}^n(s)\| \leq \frac{\kappa}{2}$ for some $s \in [0,T]$, we can also deduce (122) using a similar argument as we establish inequality (116) in the proof for Lemma 3.

In view of (122), we deduce that there exists $u_0 > 0$ such that

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{\boldsymbol{\Xi}^n(0)\in\Omega} \mathbb{E}[\exp(u_0(\|n^{1/2}\Delta\boldsymbol{\tau}^n(t_0)\| - \|n^{1/2}\Delta\boldsymbol{\tau}^n(0)\|)^+) \mid \boldsymbol{\Xi}^n(0)] \\
\leq &\limsup_{n\to\infty} \sup_{\boldsymbol{\Xi}^n(0)\in\Omega} \mathbb{E}[\exp(u_0\|n^{1/2}\Delta\boldsymbol{\tau}^n(t_0)\|) \mid \boldsymbol{\Xi}^n(0)] \\
\leq &\limsup_{n\to\infty} \sup_{\boldsymbol{\Xi}^n(0)\in\Omega} \mathbb{E}[\exp(u_0(\tfrac{\kappa}{2} + \tfrac{1}{\min_j \mu_j^n} n^{-1/2}\|\boldsymbol{Z}^n(t_0)\|))|\boldsymbol{\Xi}^n(0)] \\
< &+\infty,
\end{aligned} \tag{123}$$

where the last inequality follows from (129) in Lemma (4) (See Appendix J). Similarly, there exists $u_0 > 0$, such that

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{\boldsymbol{\Xi}^n(0)\in\Omega} \mathbb{E}[(\|n^{1/2}\Delta\boldsymbol{\tau}^n(t_0)\| - \|n^{1/2}\Delta\boldsymbol{\tau}^n(0)\|)^2 \\
&\qquad\qquad \exp(u(\|n^{1/2}\Delta\boldsymbol{\tau}^n(t_0)\| - \|n^{1/2}\Delta\boldsymbol{\tau}^n(t_0)\|)^+) \mid \boldsymbol{\Xi}^n(0)] \\
\leq &\limsup_{n\to\infty} \sup_{\boldsymbol{\Xi}^n(0)\in\Omega} \mathbb{E}[(\max\{n^{1/2}\Delta\boldsymbol{\tau}^n(0), \tfrac{\kappa}{2} + \tfrac{1}{\min_j \mu_j^n} n^{-1/2}\|\boldsymbol{Z}^n(t_0)\|\})^2 \\
&\qquad\qquad \exp(u_0(\tfrac{\kappa}{2} + \tfrac{1}{\min_j \mu_j^n} n^{-1/2}\|\boldsymbol{Z}^n(t_0)\|))|\boldsymbol{\Xi}^n(0)] \\
< &+\infty,
\end{aligned} \tag{124}$$

where the last inequality follows from (130) in Lemma (4). We have thus proved (119), and thus (58) in Proposition 4.

It remains to show that $V(\cdot)$ is a Lyapunov function with drift size parameter -1, drift term parameter t_0, and exception parameter κ for $\boldsymbol{\Xi}$, or equivalently, to prove condition (56) for $\gamma = 1$. Because $V(\boldsymbol{\Xi}^n(t))$ and $n^{-1/2}\|\Delta\boldsymbol{\tau}^n(t_0)\|$ only differs by almost a constant, proving (56) is equivalent to proving the same condition for $\|n^{1/2}\Delta\boldsymbol{\tau}^n(t)\|$ for some positive constant γ. To that end, we choose t_0 as (121) and get

$$\begin{aligned}
&\sup_{\|n^{1/2}\Delta\boldsymbol{\tau}^n(0)\|>\kappa}\{\mathbb{E}[\|n^{1/2}\Delta\boldsymbol{\tau}^n(t_0)\| \mid \|n^{1/2}\Delta\boldsymbol{\tau}^n(0)\|\} \\
\leq &\sup_{\|n^{1/2}\Delta\boldsymbol{\tau}^n(0)\|>\kappa}\{\mathbb{E}[\tfrac{\kappa}{2} + \tfrac{1}{\min_j \mu_j^n} n^{-1/2}\|\boldsymbol{Z}^n(t_0)\| \mid \|n^{1/2}\Delta\boldsymbol{\tau}^n(0)\|]\} - \kappa \\
\leq &\, c' - \tfrac{\kappa}{2}
\end{aligned} \tag{125}$$

for some constant $c' > 0$. In (125), the first inequality follows from inequality (122) and that $\|n^{1/2}\Delta\boldsymbol{\tau}^n(0)\| > \kappa$, and the second inequality follows from (128) in Lemma 4 that $n^{-1/2}\|\boldsymbol{Z}^n(t_0)\|$ is uniformly upper bounded. By choosing a sufficiently large κ, we can have $c' - \frac{\kappa}{2} < -1$, which proves that $V(\cdot)$ is a Lyapunov function with drift size parameter -1. ■

Appendix I: Proof of Theorem 5

Proof. By Proposition 4, $V(\cdot)$ is a Lyapunov function with parameter -1, t_0, and κ. Moreover, the second inequality in (58) implies that there exists u_0, such that $u_0 L_2(u_0, t_0, n) < 1$ for all sufficiently large n. Thus, both conditions of Theorem 6 in Gamarnik and Zeevi (2006) are satisfied for all sufficiently large n. We then invoke their Theorem 6 and deduce that $1 - u_0/2 > 0$ and the following inequality holds for all sufficiently n,

$$\Pr_{\boldsymbol{\pi}^n}(\|\boldsymbol{Q}^n(0)\|_T > s) \leq (1 - u_0/2)^{-1} L_1(u_0, t_0, n) \exp(-u_0(s - \kappa)). \tag{126}$$

By the above inequality and the inequality in (58), we have

$$\Pr_{\boldsymbol{\pi}^n}(\|\boldsymbol{Q}^n(0)\|_T > s) \leq H_1 \exp(-h_1 s), \tag{127}$$

for properly selected constants H_1 and h_1. Inequality (127) implies uniform tightness of the sequence of distributions $(\boldsymbol{\pi}^n)$. The rest of the proof follows exactly as in Theorem 8 of Gamarnik and Zeevi (2006). ■

Appendix J: Lemma 4 and its Proof

The following Lemma was used in both Lemma 3 and Proposition 4.

Lemma 4 *There exists a constant $u_0 > 0$, such that the following inequalities hold for all fixed $t_0 \geq 0$,*

$$\limsup_{n\to\infty} \sup_{\|\Xi^n(0) - \boldsymbol{\vartheta}\| > \kappa} n^{-1/2} \mathbb{E}[\|\boldsymbol{Z}^n\|_{t_0} | \Xi^n(0)] < \infty, \tag{128}$$

$$\limsup_{n\to\infty} \sup_{\Xi^n(0)\in\Omega} \mathbb{E}[\exp(n^{-1/2} u_0 \|\boldsymbol{Z}^n\|_{t_0}) | \Xi^n(0)] < \infty, \tag{129}$$

$$\limsup_{n\to\infty} \sup_{\Xi^n(0)\in\Omega} \mathbb{E}[\|\boldsymbol{Z}^n\|_{t_0}^2 \exp(n^{-1/2} u_0 \|\boldsymbol{Z}^n\|_{t_0}) | \Xi^n(0)] < \infty, \tag{130}$$

where $\Xi^n(0)$ gives the initial state of the Markovian process, and $\boldsymbol{Z}^n(t)$ is a J-dimensional centered process defined in (32).

Proof. Using the argument provided at the beginning of the proof for Lemma A.1 in Gamarnik and Zeevi (2006), inequality (129) implies (128) and (130). To prove (129), define $A_j^n(t) := \int_0^t p_j(\boldsymbol{X}^n(s) \circ (n\boldsymbol{\mu}^n)^{-1}) ds$. Let $S_j^*(t)$ denote the cumulative number of customers that have completed service at the j^{th} service provider up to time t,

By change of the time variables, we can derive the following bound for $n^{-1/2}\|Z_j^n\|_{t_0}$,

$$\begin{aligned}
&n^{-1/2}\|Z_j^n\|_{t_0} \\
&\leq \|n^{-1/2}(N(nt) - nt)\|_{A_j^n(t_0)} + \|n^{-1/2}(n\mu_j^n t - S_j^n(t))\|_{W_j^n(t_0)} \\
&= \|n^{-1/2}(N(t) - t)\|_{nA_j^n(t_0)} + \|n^{-1/2}(t - S_j^n(\tfrac{t}{n\mu_j^n}))\|_{n\mu_j^n W_j^n(t_0)} \\
&\leq \|n^{-1/2}(N(t) - t)\|_{nt_0} + \|n^{-1/2}(t - S_j^n(\tfrac{t}{n\mu_j^n}))\|_{2n\mu_j t_0} \\
&\leq n^{-1/2}\|N(t) - (t + B_j(t))\|_{nt_0} + n^{-1/2}\|B_j\|_{2n\mu_j t_0} \\
&\quad + n^{-1/2}\|S_j^n(t) - (t + B_j'(t))\|_{2n\mu_j t_0} + n^{-1/2}\|B_j'(t)\|_{2n\mu_j t_0}
\end{aligned} \tag{131}$$

where the second inequality follows from $A_j^n(t_0) \leq t_0$, $W_j^n(t) \leq t$, and $\mu_j^n < 2\mu_j$ for a sufficiently large n, $\boldsymbol{B} = (B_j)$ and $\boldsymbol{B}' = (B_j')$ denote two independent J-dimensional standard Brownian motions.

We next derive the tail bounds for each term at the RHS of (131). Using standard bounds for Brownian motion, we can bound the following two terms with constants $c_1, c_2 > 0$ which depend on t_0 but not on n,

$$\begin{aligned} \Pr(\|B_j\|_{nt_0} > \tfrac{1}{4}a\sqrt{n}) &= c_1\exp(-c_2a^2) \\ \Pr(\|B'_j\|_{nt_0} > \tfrac{1}{4}a\sqrt{n}) &= c_1\exp(-c_2a^2). \end{aligned} \tag{132}$$

Using the functional strong approximation theorem (FSAT) (Theorem 5.14 and Remark 5.17 in Chen and Yao (2001)), we may upper bound the tail probability of the other two terms in (131) with constants $c_3, c_4 > 0$ as follows:

$$\begin{aligned} \Pr(n^{-1/2}\|N(t) - (t + B_j(t))\|_{nt_0} \ge \tfrac{1}{4}a) &\le n^{c_3}\exp(-c_4an^{-1/2}) \\ \Pr(n^{-1/2}\|S_j^n(t) - (t + B'_j(t))\|_{2n\mu_jt_0} \ge \tfrac{1}{4}a) &\le n^{c_3}\exp(-c_4an^{-1/2}) \end{aligned} \tag{133}$$

(131), (132), and (133) together imply that

$$\Pr(n^{-1/2}\|Z_j^n\|_{t_0} \ge a) \le 2c_1\exp(-c_2a^2) + 2n^{c_3}\exp(-c_4a\sqrt{n}). \tag{134}$$

We can then upper bound the expectation $\mathbb{E}[\exp(n^{-1/2}u_0\|\boldsymbol{Z}^n\|_{t_0})|\boldsymbol{\Xi}^n(0)]$ for all sufficiently large n and initial state $\boldsymbol{\Xi}^n(0)$ using the tail probability bounds,

$$\begin{aligned} &\mathbb{E}[\exp(n^{-1/2}u_0\|\boldsymbol{Z}^n\|_{t_0})|\boldsymbol{\Xi}^n(0)] \\ &\le 2 + \textstyle\int_2^\infty \Pr\left(\exp(n^{-1/2}u_0\|\boldsymbol{Z}^n\|_{t_0}) > a\right)da \\ &= 2 + \textstyle\int_2^\infty \Pr\left(\exp(n^{-1/2}\|\boldsymbol{Z}^n\|_{t_0}) > \frac{\log x}{u_0}\right)dx \\ &\le 2 + \textstyle\int_2^\infty 2c_1\exp(-c_2\frac{\log^2 x}{u_0^2})dx + \int_2^\infty 2n^{c_3}\exp(-c_4\frac{\log x}{u_0}n^{-1/2})dx \\ &< 2M, \end{aligned} \tag{135}$$

where the second inequality follows from (134) by replacing a with $\frac{\log x}{u_0}$, and the last inequality follows from the fact that both integrals can be uniformly upper bounded by a constant $M > 0$ for sufficiently large n. Thus we have proved inequality (129). ■

Appendix K: Proof of Corollary 2

The proof is mostly similar to that of Theorem 4, but differs in two places: (1) the derivative of $Q_j^n(t)$ includes an extra term $-dX_j(t)$, which represents the aggregate reneging rate at time t; (2) the waiting time is no longer linear in $X_j(t)$ but has to be computed using equation (59). We will prove the theorem by highlighting the parts due to the above differences.

We next prove that by restricting the process to stay inside the bounded domain $\Omega(\kappa)$, the bounded process $\{\boldsymbol{Q}^{\kappa,n}(t)|0 \le t \le T\}$ weakly converges to $\{\boldsymbol{Y}^\kappa(t)|0 \le t \le T\}$. The rest of the proof, including Lemma 3, follows the same routine as in the proof for Theorem 4 and we will not repeat them.

We first express $Q_j^{\kappa,n}(t)$ in a similar way to (50) as follows:

$$\begin{aligned} Q_j^{\kappa,n}(t) &= Q_j^{\kappa,n}(0) + n^{-1/2}N\left(n\textstyle\int_0^t p_j(\boldsymbol{\tau}^{\kappa,n}(s))ds\right) - n^{-1/2}N\left(\int_0^t dX_j^{\kappa,n}(s)ds\right) - n^{-1/2}S_j^{\kappa,n}(t) \\ &\quad n^{-1/2}L_j^{\kappa,n}(t) - n^{-1/2}U_j^{\kappa,n}(t) \\ &= \underbrace{Q_j^{\kappa,n}(0)}_{\text{(A.1)}} + \\ &\quad \underbrace{\int_0^t \left(n^{1/2}\left(p_j(\boldsymbol{\tau}^{\kappa,n}(s)) - \mu_j^n - n^{-1}dX_j^{\kappa,n}(s)\right) - ((\sum_i \frac{R_{ji}^*}{(1+\tau_i^n d)\mu_i^n} - d)(Q_i^{\kappa,n}(s) - \vartheta_i) - \theta_j)\right)ds}_{\text{(A.2)}} \\ &\quad + \underbrace{n^{-1/2}Z_j^{\kappa,n}(t)}_{\text{(A.3)}} + \textstyle\int_0^t \left((\sum_i \frac{R_{ji}^*}{(1+\tau_i^n d)\mu_i^n} - d)(Q_i^{\kappa,n}(s) - \vartheta_i) - \theta_j\right)ds + \frac{1}{\sqrt{n}}L_j^{\kappa,n}(t) - \frac{1}{\sqrt{n}}U_j^{\kappa,n}(t), \end{aligned} \tag{136}$$

where the additional superscript κ represents that the corresponding process has a domain $\Omega(\kappa)$. Note that the centered process $\boldsymbol{Z}^{\kappa,n} := (Z_j^{\kappa,n})$ has included an extra term for the reneging customers, which has the expression

$$\begin{aligned} Z_j^{\kappa,n}(t) := & \Big(N(\textstyle\int_0^t np_j(\boldsymbol{\tau}^{\kappa,n}(s))) - \int_0^t np_j(\boldsymbol{\tau}^{\kappa,n}(s))ds\Big) \\ & + \big(n\mu_j^n t - S_j^{\kappa,n}(t)\big) - \Big(N(\textstyle\int_0^t dX_j^{\kappa,n}(s)ds) - \int_0^t dX_j^{\kappa,n}(s)ds\Big) \end{aligned} \tag{137}$$

We next analyze the terms labeled as (A.1)-(A.3) in (136).

1. Our assumption of the initial value implies that (A.1) $\Rightarrow \boldsymbol{Y}(0)$.
2. Since τ_j has to be computed using (59), the expression for $\Delta\boldsymbol{\tau}^{\kappa,n}$ will be

$$\begin{aligned} \Delta\boldsymbol{\tau}^{\kappa,n}(s) &:= \boldsymbol{\tau}^{\kappa,n}(s) - \boldsymbol{\tau}^* \\ &= \tfrac{1}{d}\log(1 + (n^{1/2}\boldsymbol{Q}^{\kappa,n}(s) + \boldsymbol{X}^*) \circ (n\boldsymbol{\mu}^n)^{-1}) - \tfrac{1}{d}\log(1 + \boldsymbol{X}^{n,*} \circ (n\boldsymbol{\mu}^n)^{-1}). \end{aligned} \tag{138}$$

It is not difficult to show that $n^{\frac{1}{2}}\|\Delta\boldsymbol{\tau}^{\kappa,n}\|_t$ is uniformly bounded and thus it suffices to expand the Taylor series of $n^{1/2}\left(p_j(\boldsymbol{\tau}^{n,*} + \Delta\boldsymbol{\tau}^{\kappa,n}(s)) - p_j(\boldsymbol{\tau}^{n,*})\right)$ till its first-order term. Some basic algebra leads to

$$n^{1/2}\left(p_j(\boldsymbol{\tau}^{n,*} + \Delta\boldsymbol{\tau}^{\kappa,n}(s)) - p_j(\boldsymbol{\tau}^{n,*})\right) \to \sum_i \frac{Q_i^{\kappa,n}(s) - \vartheta_i}{(1+\tau_j^* d)\mu_i^n} R_{ji}^* \tag{139}$$

Thus, by our definition of θ_j and ϑ_j, we have

$$\begin{aligned} & n^{1/2}\left(p_j(\boldsymbol{\tau}^n(s)) - \mu_j^n - n^{-1}d\boldsymbol{X}^{\kappa,n}(s)\right) \\ &= n^{1/2}\left(p_j(\boldsymbol{\tau}^{n,*} + \Delta\boldsymbol{\tau}^{\kappa,n}) - p_j(\boldsymbol{\tau}^{n,*})\right) + n^{1/2}\left(p_j(\boldsymbol{\tau}^{n,*}) - \mu_j^n - n^{-1}d\boldsymbol{X}^{n,*}\right) \\ &\quad + n^{1/2}\left(n^{-1}d\boldsymbol{X}^{n,*} - n^{-1}d\boldsymbol{X}^*\right) - n^{1/2}\left(n^{-1}d\boldsymbol{X}^{\kappa,n}(s) - n^{-1}d\boldsymbol{X}^*\right) \\ &\to \textstyle\sum_i \frac{Q_i^{\kappa,n}(s)-\vartheta_i}{(1+\tau_j^* d)\mu_i^n} R_{ji}^* - \theta_j + d\vartheta_j - dQ_j^{\kappa,n}(s) \end{aligned} \tag{140}$$

The above convergence leads to that (A.2)→ 0 uniformly over any compact set.

3. $n^{-1/2}Z^{\kappa,n}(t)$ is the sum of three centered processes. We have shown in the proof of Theorem 4 that the sum of the first two terms converges to $\boldsymbol{\Sigma}\boldsymbol{B}(t)$ with $\boldsymbol{\Sigma}$ a diagonal matrix and $\Sigma_{jj} = \sqrt{(1+\omega_j)^2\mu_j}$, respectively. Since $\frac{1}{n}\int_0^t dX_j^{\kappa,n}(s)ds \to \frac{1}{n}dX_j^* t = (\exp(\tau_j^* d) - 1)\mu_j t$ uniformly on any compact set $t \in [0,T]$, and $\frac{1}{n}\int_0^t dX_j^{\kappa,n}(s)ds$ is a non-decreasing process in t, we may invoke the random time-change theorem and FCLT to prove that

$$n^{-1/2}\left(N(\int_0^t dX_j^{\kappa,n}(s)ds) - \int_0^t dX_j^{\kappa,n}(s)ds\right) \Rightarrow B_j^D(t). \tag{141}$$

where $B_j^D(t)$ is a Brownian motion whose covariance matrix is a diagonal matrix and the j^{th} entry of its diagonal is given by $(\exp(\tau_j^* d) - 1)\mu_j$. Since $n^{-1/2}\boldsymbol{Z}^{\kappa,n}(t)$ is the sum of three independent Brownian processes, we deduce that

$$n^{-1/2}\boldsymbol{Z}^{\kappa,n}(t) \Rightarrow \boldsymbol{\Sigma}^R\boldsymbol{B}(t). \tag{142}$$

■