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Tight Bounds for The Price of Fairness

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A central decision maker (CDM), who seeks an efficient allocation of scarce resources among a finite number of players, often has to incorporate fairness criteria to avoid unfair outcomes. Indeed, the Price of Fairness (POF), a term coined in Bertsimas et al. (2011), refers to the efficiency loss due to the incorporation of fairness criteria into the allocation method. Quantifying the POF would help the CDM strike an appropriate balance between efficiency and fairness. In this paper we improve upon existing results in the literature, by providing tight bounds for the POF for the proportional fairness criterion for any n, when the maximum achievable utilities of the players are equal or are not equal. Further, while Bertsimas et al. (2011) have already derived a tight bound for the max-min fairness criterion for the case that all players have equal maximum achievable utilities, we also provide a tight bound in scenarios where these utilities are not equal. Finally, we investigate the sensitivity of our bounds and bounds of Bertsimas et al. (2011) for the POF to the variability of the maximum achievable utilities.

Key words: price of fairness, resource allocation, proportional fairness, max-min fairness

1. Introduction

In this paper, we consider a problem facing a central decision maker (CDM) who needs to allocate scarce resources among finitely many players. There are many real-life applications wherein the CDM cannot use the most efficient allocation due to a fairness constraint. Instead, the CDM often has to allocate resources while recognizing the tradeoffs between efficiency and fairness (Zenios et al. 2000, Bertsimas et al. 2012). In the literature, two fairness notions have been widely used: Proportional Fairness (PF) and Max-Min Fairness (MMF). Our goal is to measure the relative system efficiency loss resulting from the implementation of these fairness criteria.

An illustrative example of the efficiency-fairness trade-off is the allocation of deceased-donor kidneys. Kidney transplantation is a desired treatment for individuals with end-stage renal disease (ESRD). However, the reality is that the demand for kidneys far exceeds the available supply. To illustrate, in 2021, 43,617 new patients registered to the kidney waiting list, while only 25,490 transplants were performed that year[1]. This significant discrepancy between the number of patients in need and the available transplants highlights the significance of allocating deceased-donor kidneys in an efficient and fair manner. Efficiency, in this context, refers to maximizing the overall utility, usually measured by the total quality-adjusted life-years (QALY). The problem of maximizing the sum of the utilities among all candidates on the waitlist has been extensively studied in the operations research literature; see, e.g., Derman et al. (1972), Righter (1989) and David and Yechiali (1995). However, a policy solely focused on maximizing total QALYs may inadvertently disadvantage patient groups with lower expected QALYs gained from transplant. If the policy maker seeks to minimize such inequities, they must contend with the trade-off of potentially reducing the overall number of life-years saved, e.g., Ata et al. (2021), Su and Zenios (2006), Bertsimas et al. (2013).

Similar trade-offs between efficiency and fairness naturally emerge in many other applications. For example, in air traffic flow management, the CDM needs to fairly allocate limited air space to different airlines (e.g., Barnhart et al. 2012, Vossen et al. 2003, Jiang et al. 2022, Nguyen et al. 2021). In congested airports, the CDM has to fairly allocate runways and aprons to different airlines (Fairbrother et al. 2020). Other equitable resource allocation problems arise in finance, where simultaneous trading with multiple accounts is carried out and the CDM's goal is to treat all clients fairly while maximizing collective interests (O'Cinneide et al. 2006), water allocation, where a CDM attempts to balance unsatisfied demand of all consumers (Udías et al. 2012), medical funds allocation, where equity across different regions and populations is a major concern (Earnshaw et al. 2007), spectrum allocation in a communication system where a common frequency band is shared by multiple users (Ye 2014), and kanban allocation in production systems, where both minimizing the rate of lost sales and balancing lost sales across product types are under consideration (Ryan and Vorasayan 2005). In addition to resource allocation problems, efficiency-fairness tradeoffs can arise in classical optimization problems such as location, vehicle routing, transportation and scheduling, as elaborated in the excellent survey papers by Karsu and Morton (2015) and Luss (1999).

Naturally, quantifying the efficiency loss resulting from the incorporation of different fairness criteria is essential for the CDM when facing resource allocation challenges. Bertsimas, Farias, and Trichakis (2011) (BFT (2011)) introduced the notion of the price of fairness (POF), which is defined as the relative efficiency loss, compared to the utility-maximization solution (i.e., utilitarian solution), of a fair solution. They quantified the POF for two different fairness notions, PF and MMF. That is, they provided upper bounds of the POF for both the PF and the MMF cases, which we refer to as the BFT bounds. When all players have equal maximum achievable utilities, the BFT bound for the PF criterion is tight when the number of players, n, is a square of an integer, and the BFT bound for the MMF criterion is tight for an arbitrary number of players.

The assumption that players have equal maximum achievable utilities is appropriate in settings in which the utility functions merely represent the players' preferences among the various resource allocation options. These preferences are invariant to positive affine transformations of the corresponding utility functions, and thus, permit the assumption that in such settings, the maximum achievable utilities of all players are equal. However, there are many applications in which the utility functions have some intrinsic values, which facilitates interpersonal comparisons of utilities. For instance, in the context of organ transplantation, as previously discussed, the utility of each patient is commonly quantified by QALYs (Su and Zenios 2005, Zenios et al. 2000). In air traffic flow management, the disutility of an airline is measured by the total time delays (Barnhart et al. 2012), and in most business applications, a firm's utility can be measured by a monetary value. In all such examples, the utilities of players cannot be normalized and the assumption of equal maximum achievable utilities cannot be made without loss of generality. Indeed, BFT (2011) have investigated resource allocation problems without the assumption of equal maximum achievable utilities, and have derived bounds for the POF both for the PF and MMF criteria. However, both bounds are not tight[2]. By contrast, in this paper we derive tight bounds for the POF both for the PF and MMF criteria, when the maximum achievable utilities are not necessarily equal.

We investigate the gap between our bound and the BFT bound for the proportional POF when the maximum achievable utilities are equal. Clearly, this gap is always non-negative, and we prove that it achieves a local maximum when n is a product of two consecutive integers, e.g., $n = 2, 6, 12$, etc. Both bounds increase as a function of n, and as noted by BFT (2011), for a small number of players, the price of proportional fairness is small. Indeed, for example, for $n = 2$, which corresponds to Nash original two-players bargaining problem, the price of fairness according to our bound is at most 6.7%. By comparison, we note that according to the BFT bound, the price of proportional fairness for $n = 2$ is at most 8.6%. We further show that for a fixed number of players, both for PF and MMF, our bounds and the BFT bounds increase when the variability of the maximum achievable utilities increases. However, the BFT bounds increase at a faster rate. The tighter bounds and the additional insights we derive into the price of fairness should assist a CDM to better evaluate the tradeoff between efficiency and fairness.

Our paper makes the following contributions:

1. We are the first to derive tight upper bounds for the POF applicable to both PF and MMF criteria under general conditions, that is, for an arbitrary number of players and when the players' maximum achievable utilities are possibly unequal. Our results complement those by BFT (2011) by tightening their bounds for the POF, particularly in the unequal maximum achievable utilities case.
2. The tight bounds provide robust insights that should assist a CDM to more effectively evaluate the tradeoff between efficiency and fairness. Our bounds confirm and improve upon results derived by BFT (2011) that the POF is likely to be small for a small number of players. When players have different maximum achievable utilities, the POF is relatively larger when the

distribution of the maximum achievable utilities has a larger variance. In fact, the POF is at its peak when this distribution is such that one player has a large maximum achievable utility, while all others have a very small maximum achievable utilities.

3. A growing body of research, exemplified by studies such as (Hasankhani and Khademi 2022, Ma et al. 2023, Agnetis et al. 2019), has been concerned with calculating the POF for special problem instances. Our tight bounds for the POF in general settings could serve not only as a cap but also as a benchmark for the POF estimates in settings previously studied in the literature. This underscores the broader applicability and relevance of our findings within the literature on POF across various operational frameworks.

2. Fairness

2.1. Related Literature

There are two stream of literature closely related to our study of the POF. The first stream is concerned with the various fairness criteria. It is generally understood that different equity/fairness criteria may be required in different contexts/applications (see, e.g., Sen 1997, Young 1995), and that no universal criterion of fairness can be applied in all settings. Nevertheless, there are several widely accepted criteria for fairness.

A simple and common fairness criterion, based on the Rawlsian principle (Rawls 2004), is the MMF criterion. According to this criterion we seek a solution which is the lexicographically largest vector, whose elements are either the allocations to the different players or the performance function values corresponding to the different activities, which are arranged in a non-increasing order.

Some studies handle fairness using an inequality index, which is a function that maps a resource allocation instance to a scalar value representing the level of inequality. For example, Kozanidis (2009) uses the difference between the upper bound and the lower bound of outcomes as the inequality index, and Turkcan et al. (2011) uses variance to measure fairness.

Another approach to achieve fairness optimizes an objective function which is some aggregation of the allocations to the different players. Indeed, proportional fairness (PF), which we study in this paper, is achieved by maximizing the sum of the logarithms of the utility outcomes corresponding to the allocations to the players. It is a generalization of the Nash bargaining solution, see, e.g., Nash Jr (1950), which has an axiomatic basis as we further elaborate in the sequel. Finally for a classification of the extensive literature concerned with fairness in terms of the fairness criteria they employ, such the Rawlsian principle, its lexicographic extension or an aggregation function, see the survey by Karsu and Morton (2015).

The second stream of literature related to our study of the POF focuses on the assessment of the POF in particular contexts. Recent papers in this stream include, for example, Hasankhani and

Khademi (2022), who studied the US heart transplant system and, using a fluid model, have quantified the POF for both the PF and MMF fairness criteria. They have shown, for example, that consistent with known theoretical results the price of PF is smaller than that of MMF. Liu and Salari (2022) calculated the POF for a facility location problem, considering disutility as an aggregate of transit time and queueing delays. Agnetis et al. (2019) were the first to investigate the POF in scheduling problems, providing tight bounds within their defined context. Zhang et al. (2020) extended this exploration to a two-agent scheduling game variant, specifically examining scenarios where one of the two players has exactly two jobs. As noted above, our tight bounds for the POF for the PF and MMF criteria, and our related investigation about the sensitivity of the POF to the distribution of the maximum achievable utilities could be helpful in future investigations into the POF in different contexts and application areas.

2.2. Fairness Notions

We quantify the price of fairness for two fairness criteria - PF and MMF.

2.2.1. Proportional fairness Under PF, the preferred allocation, $u^{PF}(U) = (u_1^{PF}(U), u_2^{PF}(U), \cdots, u_n^{PF}(U))$, from a utility set U, is such that for any other feasible allocation of utilities u, the aggregate of proportional changes is zero or negative, i.e.,

$$\sum_{i=1}^{n} \frac{u_i - u_i^{PF}(U)}{u_i^{PF}(U)} \leq 0, \qquad \text{for any } u \in U. \tag{1}$$

For a convex utility set U, $u^{PF}(U)$ is a generalization on the Nash bargaining solution for n players, and can be obtained by maximizing the product of the players' utilities, i.e.,

$$u^{PF}(U) := \underset{(u_1, u_2, \cdots, u_n) \in U}{\arg\max} \prod_{i=1}^{n} u_i.$$

Equivalently, $u^{PF}(U)$ can be derived by maximizing the logarithmic transformation of the above objective function, i.e.,

$$u^{PF}(U) := \underset{(u_1, u_2, \cdots, u_n) \in U}{\arg\max} \sum_{i=1}^{n} \log u_i. \tag{2}$$

Note that (1) is the necessary and sufficient condition for the optimality of $u^{PF}(U)$ for the optimization problem associated with (2). As mentioned above, such a solution is a generalization of the Nash bargaining solution. It satisfies four axioms: Pareto optimality, symmetry, affine invariance and independence of irrelevant alternatives. Pareto optimality avoids a waste of resources. Symmetry guarantees that the solution does not distinguish between the players if the model does not distinguish between them. By the third axiom, the allocation is invariant to scaling of utilities, and thus would not be affected if different measurement units of utility are used by players. The fourth axiom implies that the preferred allocation remains unchanged when inferior allocations are removed, see, e.g., Nash Jr (1950) and Roth (1979a).

The first application of PF was in the telecommunications field (Mazumdar et al. 1991), and the term PF was first coined by Kelly et al. (1998).

2.2.2. Max-min fairness MMF is motivated by the Kalai-Smorodinsky (KS) solution for two-person bargaining problems, axiomatically characterized by Kalai and Smorodinsky (1975). The KS solution is the unique solution satisfying the axioms of Pareto optimality, symmetry, invariance with respect to affine transformations of utility, and monotonicity. According to the KS solution, the players obtain the largest possible equal fraction of their respective maximum achievable utilities. However, Roth (1979b) has shown that in n-person bargaining problems, such a solution may not satisfy Pareto optimality and that there does not exist a solution satisfying all the above axioms. Imai (1983) has modified the KS solution to n-person bargaining problems by proposing a weaker set of axioms. Namely, he proposes a solution which satisfies Pareto optimality, symmetry and invariance under linear utility transformation. But, instead of the monotonicity axiom, originally proposed by Kalai and Smorodinsky (1975), he requires the axioms of independence of irrelevant alternatives other than ideal point and individual monotonicity.

The axiom of independence of irrelevant alternatives other than ideal point is a less stringent requirement compared to the axiom of independence of irrelevant alternatives, which is satisfied by the Nash bargaining solution. It stipulates that the solution remains unchanged if the alternatives that have been removed do not alter the maximum achievable utilities by all players. The other axiom, individual monotonicity, requires that if the utility set expands in a manner which keeps the projection of the utility set onto the $N\backslash\{i\}$-dimensional space unchanged, then the utility of the i-th player must increase.

Imai (1983) has proven that when all players have equal maximum achievable utility, his solution coincides with the MMF solution (referred to as lex-max-min in his paper), which, according to BFT (2011), maximizes the ratios of players' utilities to their maximum achievable utilities in a lexicographical manner. That means, the CDM first maximizes the minimum ratio of the players' utilities to their respective maximum achievable utilities. Subsequently, the CDM maximizes the second smallest ratio, the third smallest ratio, and so on. Thus, for two-person problems, the MMF solution coincides with the KS solution.

2.3. The Price of Fairness

The utilitarian solution, which maximizes the the sum of the utilities to all the players, is viewed as a measure for system efficiency. Naturally, implementing a fair solution, instead of a utilitarian solution, will reduce system efficiency. To quantify the relative loss incurred by adopting a fair solution over the utilitarian solution, we employ the concept of the POF as introduced by BFT (2011).

Formally, consider a resource allocation problem with n players, and let $U :\subseteq \mathbf{R}^n$ denote the set of all possible utility vectors of the players. Let u_i denote the utility of player i. We assume U to be convex and compact, which is a common assumption in the literature (See BFT (2011)). We also assume that U is monotone, that is:

Definition 1. A set U is **monotone** if for any $u \in U$ and $0 \le v \le u$, we have $v \in U$.

2.3.1. Utilitarian Solution A utilitarian solution is an allocation based on classical utilitarianism, which maximizes the sum of utilities of all players. Thus, the utilitarian solution, denoted by $u^*(U) := (u_1^*(U), \ldots, u_n^*(U))$, is an optimal solution to the following problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^n u_i, \\ \text{s.t.} \quad & (u_1, u_2, \cdots, u_n) \in U. \end{aligned} \tag{3}$$

The sum of utilities to the players is often used as a measure of system efficiency. Thus, if the social planner aims to maximize efficiency without fairness considerations, the utilitarian solution should be used.

2.3.2. Fairness Solution As previously mentioned, in many applications it would be inappropriate to implement solutions based exclusively on system efficiency. In these circumstances, the central decision maker will choose an allocation scheme based on some fairness criteria, trying to balance both efficiency and fairness.

There is a large literature concerned with various fairness notions. We consider two notions in this paper, PF and MMF. We will explain them in detail in the next section. We denote the allocation scheme with a set function $u^{fair} : 2^{\mathbb{R}_n^+} \to \mathbb{R}_n^+$, where the superscript, fair, could be PF or MMF to indicate whether we use PF or MMF.

2.3.3. The Price of Fairness We use the notion of the price of fairness, introduced by BFT (2011) and denoted $POF(U; fair)$, to measure the relative loss resulting from an implementation of a fair solution rather than the utilitarian solution, i.e.,

$$POF(U; fair) = \frac{\sum_{i=1}^n u_i^* - \sum_{i=1}^n u_i^{fair}}{\sum_{i=1}^n u_i^*} = 1 - \frac{\sum_{i=1}^n u_i^{fair}}{\sum_{i=1}^n u_i^*}, \tag{4}$$

where u_i^*, resp., $u_i^{fair}, i = 1, \ldots, n$, denotes the utilitarian, respectively, the fair solution which is being used.

BFT (2011) have derived upper bounds for the $POF(U; fair)$ for the cases where the fair solution is either the PF solution or the MMF solution, and for the cases when the maximum achievable utilities of the players are either equal or not equal. As will be clarified in the sequel, we improve upon the bounds derived by BFT (2011), by providing tight bounds for the price of proportional fairness when the maximum achievable utilities by the players are either equal or not necessarily equal, and for the price of MMF when the maximum achievable utilities by the players are not necessarily equal.

3. Upper Bounds for The Price of Proportional Fairness

3.1. Equal Maximum Achievable Utilities

We begin by considering the case where all players have equal maximum achievable utilities in the utility set U. For simplicity and without loss of generality, we set this maximum achievable utility to one, and denote by $N := \{1, 2, \ldots, n\}$, the set of players. Consequently, $\max\{u_i | u \in U\} = 1$ for $i \in N$.

To derive an upper bound for the price of fairness (POF) in U, we employ a utility set U', $U' \supseteq U$, such that the PF solutions in U and U' coincide. Consequently, since $U \subseteq U'$, the POF with respect to U is bounded by the POF with respect to U'. Formally,

Proposition 1. *Suppose $U \subseteq \{u := (u_1, u_2, \ldots, u_n) \mid 0 \le u_i \le 1, i \in N\}$ is a convex, compact, and monotone utility set, and $\max\{u_i|u \in U\} = 1$ for $i \in N$. Then there exist $c_i \in [1/n, 1]$ ($i \in N$) such that $U \subseteq U' = \{u \mid \sum_{i=1}^n c_i u_i \le 1, 0 \le u_i \le 1, i \in N\}$, and $\max\{\sum_{i=1}^n \log(u_i)|u \in U\} = \max\{\sum_{i=1}^n \log(u_i)|u \in U'\}$. Consequently, $POF(U; PF) \le POF(U'; PF)$.*

Proof. Let v denote the optimal solution to $\max\{\sum_{i=1}^n \log(u_i)|u \in U\}$. Define $\nu(u) := \sum_{i=1}^n \log(u_i)$ and denote by $\nabla(\nu(v))$ the gradient of this function $\nu(\cdot)$ at v. Since U is convex and compact, and the objective function $\nu(\cdot)$ is concave, the necessary and sufficient optimality condition is

$$(u - v)^T \nabla(\nu(v)) = \sum_{i=1}^n \left(\frac{u_i}{v_i} - 1\right) \le 0 \Leftrightarrow \sum_{i=1}^n c_i u_i \le 1, \quad \forall\, u \in U, \tag{5}$$

where $c_i := 1/(nv_i)$. Therefore, the optimality condition implies that $U \subseteq U' := \{u \mid \sum_{i=1}^n c_i u_i \le 1, 0 \le u_i \le 1, i \in N\}$, for $c_i = 1/(nv_i)$.

We claim that v is also the optimal solution to the optimization problem to derive the PF solution with respect to U', i.e., $\max\{\sum_{i=1}^n \log(u_i)|u \in U'\}$. Indeed, since $v \in U \subseteq U'$, v is feasible to this optimization problem. Further, for all $u \in U'$, we have $\sum_{i=1}^n c_i u_i = \sum_{i=1}^n u_i/(nv_i) \le 1$. Thus, v satisfies the necessary and sufficient optimality condition (5). Therefore, v is also the PF solution with respect to U'. We thus have $u^{PF}(U) = u^{PF}(U')$.

Given that $U \subseteq U'$, we have $\sum_i u_i^*(U) \le \sum_i u_i^*(U')$, where $u_i^*(U)$ and $u_i^*(U')$ denote player i's utility at the utilitarian solution with respect to U and U', respectively. Therefore, by the definition of the POF, i.e., (4), we have $POF(U; PF) \le POF(U'; PF)$.

Finally, since $0 \le v_i \le 1$, we have $c_i = 1/(nv_i) \ge 1/n$. Let e_i denote a vector with the i-th element as one and all other elements as zero. By assumptions of U, $e_i \in U$ and thus $e_i \in U'$ for $i \in N$. Then $\sum_{i=1}^n c_i u_i \le 1 \Rightarrow c_i \le 1$. ■

By Proposition 1, we can derive an upper bound for the price of proportional fairness (PF) with respect to U, by finding this bound with respect to U', $U' = \{(u_1, u_2, \ldots, u_n)|\sum_{i=1}^n c_i u_i \le 1,\ 0 \le u_i \le$

$1,\ i \in N\}$. To that end, we need to find the utilitarian solution $u^*(U')$, which is an optimal solution to the following Problem (6):

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} u_i, && (6a)\\ \text{s.t.} \quad & 0 \le u_i \le 1, \qquad \forall i \in N, && (6b)\\ & \sum_{i=1}^{n} c_i u_i \le 1. && (6c) \end{aligned}$$

Without loss of generality, we assume that

$$c_1 \le c_2 \le \cdots \le c_n. \tag{7}$$

Following BFT (2011), we define

$$l(c) := \max\{j \mid \sum_{i=1}^{j} c_i \le 1\}, \qquad \delta(c) := \frac{1 - \sum_{i=1}^{l(c)} c_i}{c_{l(c)+1}} \in [0,1). \tag{8}$$

Then, the optimal solution, u^*, to Problem (6) is $u_1^* = u_2^* = \cdots = u_{l(c)}^* = 1, u_{l(c)+1}^* = \delta(c), u_{l(c)+2}^* = u_{l(c)+3}^* = \cdots = u_n^* = 0$, and the optimal value of Problem (6) is $l(c) + \delta(c)$.

On the other hand, the PF solution can be obtained by solving the following optimization Problem (9):

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} log(u_i), && (9a)\\ \text{s.t.} \quad & 0 \le u_i \le 1, \qquad \forall i \in N, && (9b)\\ & \sum_{i=1}^{n} c_i u_i \le 1. && (9c) \end{aligned}$$

Problem (9) has a unique optimal solution, u^{PF}, which has the following explicit form:

PROPOSITION 2. $u^{PF}(U') := (1/nc_1, 1/nc_2, \cdots, 1/nc_n)$ *is the unique optimal solution to the optimization Problem* (9)*.*

Proof. Since $c_i \ge 1/n$, we have $u_i^{PF}(U') \le 1$. So $u^{PF}(U')$ is a feasible solution to (9). For any feasible $u \in U'$, $(u - u^{PF}(U'))^T \nabla(\nu(u^{PF}(U'))) = \sum_{i=1}^{n}(u_i - u_i^{PF}(U'))/u_i^{PF}(U') = \sum_{i=1}^{n} nc_i u_i - n \le 0$, which implies the optimality of $u^{PF}(U')$. Finally, since in (9), we maximize a strictly concave function over a convex compact domain, the maximizer must be unique. ■

The utilitarian solution, u^*, and the PF solution, u^{PF}, we have derived with respect to U', can be used to find an upper bound for the price of proportional fairness by minimizing $1 - POF(U'; PF) =$

$\sum_{i=1}^n u_i^{PF}(U')/\sum_{i=1}^n u_i^*(U') := f_1(c)$, which is equivalent to the following optimization Problem (10):

$$\min_c \quad (f_1(c) :=) \frac{\sum_{i=1}^n \frac{1}{nc_i}}{l(c) + \frac{1-\sum_{i=1}^{l(c)} c_i}{c_{l(c)+1}}}, \tag{10a}$$

$$\text{s.t.} \quad \frac{1}{n} \le c_1 \le c_2 \le \cdots \le c_n \le 1, \tag{10b}$$

$$\sum_{i=1}^{l(c)} c_i \le 1, \tag{10c}$$

$$\sum_{i=1}^{l(c)+1} c_i > 1, \tag{10d}$$

where Constraints (10c), (10d) follow from the definitions of $l(c)$ and $\delta(c)$, as given in the Expression (8).

A key step towards the main result is the characterization of c^*, the optimal solution to Problem (10), which is provided by Proposition 3 below.

Proposition 3. *(1) When $n=2$, $c_1^* = \sqrt{3}-1, c_2^* = 1$.*
(2) When $n \ge 3$, $c_1^* = \cdots = c_m^* = 1/m$, $c_{m+1}^* = \cdots = c_n^* = 1$, and $\delta(c^*) = 0$.

Proof.

The proof follows from the following claims, whose proofs are provided in Appendix EC.1.

Claim 1. *Define $m := l(c^*)$. Then, $c_i^* = 1$ for $i = m+2, \cdots, n$.*

Claim 2. $c_1^* = c_2^* = \cdots = c_m^*$.

Claim 3. $c_{m+1}^* = 1$.

Claim 4. *When $m \ge 2$, we have $d^*(:= c_1^* = c_2^* = \cdots = c_m^*) = 1/m$.*

Claim 5. *When $n \ge 3$, $m \ne 1$.*

Claim 6. *When $n = 2$, $m = 1, c_1^* = \sqrt{3}-1$.*

When $n = 2$, Proposition 3 is implied by Claim 3 and 6. When $n \ge 3$, we have $m \ne 1$ by Claim 5, thus $m \ge 2$. Then by Claim 4, $d^* = 1/m$. Thus by Claims 1-3, the optimal solution to Problem (10) is $c_1^* = \cdots = c_m^* = 1/m, c_{m+1}^* = \cdots = c_n^* = 1$, which implies $\delta(c^*) = 0$. ∎

Proposition 3 provides a complete characterization of the optimal solution to Problem (10), which leads to the first main result on the upper found of the POF for the proportional fairness case.

THEOREM 1. *Consider a resource allocation problem with n players, $n \geq 2$, for which the utility set $U \subseteq [0,1]^n$ is compact and convex, and the maximum achievable utilities of all players are equal. Let $\sqrt{n} = k + \epsilon$, where $k \in \mathbb{N}$ is the integral part of $\sqrt{n}$ and $\epsilon \in [0,1)$ is the fractional part. Then, the tight bound of the price of proportional fairness solution is:*

(a) For $n = 2$,

$$POF(U;PF) \leq \frac{2-\sqrt{3}}{4}.$$

(b) For $n \geq 3$:

(1) For $0 \leq \epsilon \leq (1 + 2\sqrt{n} - \sqrt{1+4n})/2$,

$$POF(U;PF) \leq 1 - \frac{2\sqrt{n} - 1 + \frac{\epsilon^2}{k}}{n}.$$

(2) For $(1 + 2\sqrt{n} - \sqrt{1+4n})/2 < \epsilon < 1$,

$$POF(U;PF) \leq 1 - \frac{2\sqrt{n} - 1 + \frac{(1-\epsilon)^2}{k+1}}{n}.$$

Proof. (a) For $n = 2$, we have $f_1(c^*) = (2+\sqrt{3})/4$ by Proposition 3, and

$$POF(U;PF) \leq 1 - f_1(c^*) = 1 - \frac{2+\sqrt{3}}{4} = \frac{2-\sqrt{3}}{4}.$$

(b) For $n \geq 3$, by Proposition 3, the objective function in (10a) can be reformulated as

$$f_1(c^*) = \frac{\sum_{i=1}^{m} \frac{1}{nc_i^*} + \frac{n-m}{n}}{m} = \frac{1}{n}(m + \frac{n}{m} - 1). \tag{11}$$

Since this function decreases and then increases as m increases, the necessary and sufficient conditions for the optimality of m are:

$$\left.\begin{array}{l} m + \dfrac{n}{m} - 1 \leq m + 1 + \dfrac{n}{m+1} - 1 \Leftrightarrow n \leq m(m+1) \\ m + \dfrac{n}{m} - 1 \leq m - 1 + \dfrac{n}{m-1} - 1 \Leftrightarrow n \geq m(m-1) \end{array}\right\}$$
$$\Leftrightarrow \qquad \frac{\sqrt{1+4n}-1}{2} \leq m \leq \frac{\sqrt{1+4n}+1}{2}.$$

Since $(\sqrt{1+4n}+1)/2 - (\sqrt{1+4n}-1)/2 = 1$, m is contained in an interval of length 1. Also, $\sqrt{n}$ lies in this interval. Thus, since $k = \sqrt{n} - \epsilon$, m can take only two possible values, k or $k+1$.

Thus, if $(\sqrt{1+4n}-1)/2 \leq k \leq (\sqrt{1+4n}+1)/2 \Rightarrow 0 \leq \epsilon \leq (1 + 2\sqrt{n} - \sqrt{1+4n})/2$, then $f_1(c^*) = (n + k(k-1))/nk = (2\sqrt{n} - 1 + \epsilon^2/k)/n$, and

$$POF(U;PF) \leq 1 - \frac{2\sqrt{n} - 1 + \frac{\epsilon^2}{k}}{n}.$$

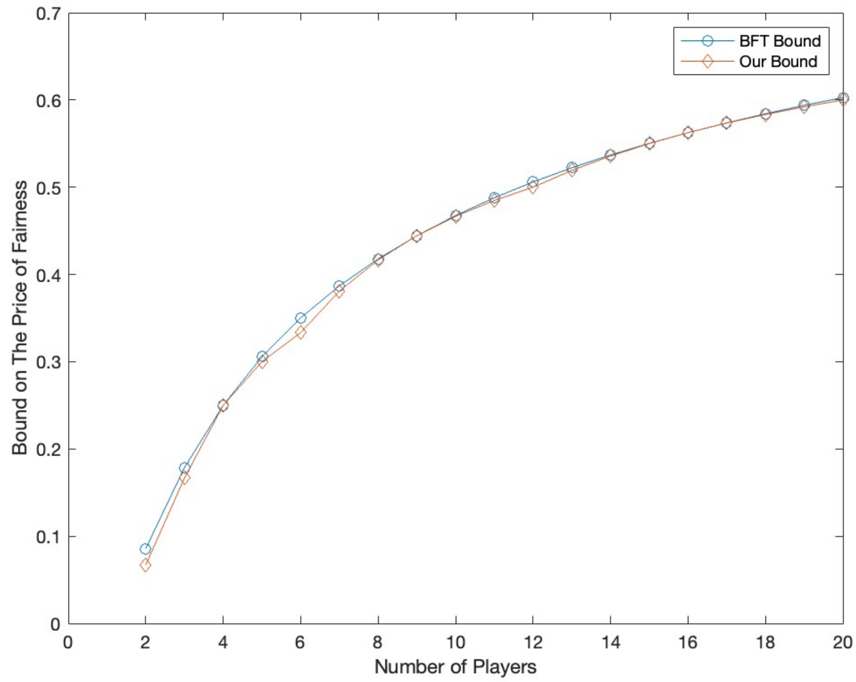


Figure 1 Upper bound of the Proportional POF as a function of the number of players.

If $(\sqrt{1+4n}-1)/2 < k+1 < (\sqrt{1+4n}+1)/2 \Rightarrow (1+2\sqrt{n}-\sqrt{1+4n})/2 < \epsilon < 1$, then $f_1(c^*) = (n+k(k+1))/(n(k+1)) = (2\sqrt{n}-1+(1-\epsilon)^2/(k+1))/n$, and

$$POF(U;PF) \leq 1 - \frac{2\sqrt{n}-1+\frac{(1-\epsilon)^2}{k+1}}{n}.$$

The above bounds are tight, as we have explicitly derived in Proposition 2 and Proposition 3 the optimal solution to Problem (10).

■

Figure 1 plots the BFT bound as well as our bound for the price of proportional fairness, $POF(U;PF)$, as a function of the number of players, n. The two bounds are quite similar which reveals that the BFT bound, in general, performs relatively well when players have equal maximum achievable utilities.

Let us further study the difference between our bound and the BFT bound, and note that for $\sqrt{n} \in \mathbb{N}$, i.e., $\epsilon = 0$, the BFT bound coincides with our bound. Let us consider the relative improvement, $\Delta(n)$, of our bound over the BFT bound:

$$\Delta(n) := \frac{\text{BFT bound} - \text{our bound}}{\text{BFT bound}} = \begin{cases} \frac{\epsilon^2}{k(n-2\sqrt{n}+1)}, & 0 \leq \epsilon \leq \frac{1+2\sqrt{n}-\sqrt{1+4n}}{2}; \\ \frac{(1-\epsilon)^2}{(k+1)(n-2\sqrt{n}+1)}, & \frac{1+2\sqrt{n}-\sqrt{1+4n}}{2} < \epsilon < 1. \end{cases}$$

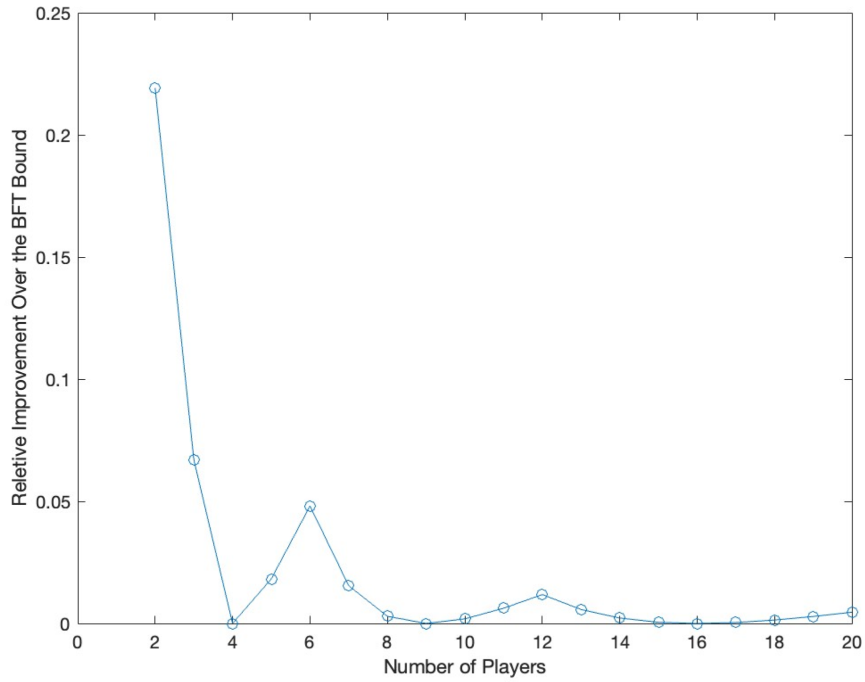


Figure 2 **The relative improvement of our upper bound of the proportional POF over the BFT bound.**

Figure 2 plots the relative improvement, $\Delta(n)$. It shows that $\Delta(n)$ reaches a local maximum when the number of players, n, is a product of two adjacent integers, that is, when $n = l(l+1)$ for some $l \in \mathbb{N}$, as in the cases when $n = 6, 12, \ldots$, etc. In Lemma 1 below, whose proof is in Appendix EC.2, we formally prove this result.

Lemma 1. *For any $a \in \mathbb{N}^+$ and $n \geq 2$, when $n \in [a^2, a(a+1)]$, $\Delta(n)$ increases with respect to n; when $n \in [a(a+1), (a+1)^2)$, $\Delta(n)$ decreases with respect to n. Thus, $\Delta(n)$ reaches a local maximum when $n = a(a+1)$.*

Finally, as BFT (2011) have noticed, the price of proportional fairness is relatively small when n is small. In particular, for $n = 2$, which corresponds to the Nash Bargaining two-player game setting, the BFT bound is 8.6%. In that regard, we note that for $n = 2$, our bound for POF is 6.7%, an improvement of 22% over the BFT bound, as shown in Figure 2.

3.2. Unequal Maximum Achievable Utilities

We next consider the case where players have unequal maximum achievable utilities. Denote the maximum achievable utility for player i by L_i. Thus, $0 \leq u_i \leq L_i$ for all $u \in U$.

Proposition 4. *Suppose $U \subseteq \{u \,|\, 0 \leq u_i \leq L_i, i \in N\}$ is a convex, compact, and monotone utility set, and $\max_{u \in U} u_i = L_i$ for $i \in N$. Then there exist $c_i \in [1/nL_i, 1/L_i]$, $i \in N$, such that $U \subseteq U' =$*

$\{u \,|\, \sum_{i=1}^n c_i u_i \le 1,\, 0 \le u_i \le L_i,\, i \in N\}$, *and* $\max\{\sum_{i=1}^n \log(u_i) \,|\, u \in U\} = \max\{\sum_{i=1}^n \log(u_i) \,|\, u \in U'\}$. *Consequently,* $POF(U;PF) \le POF(U';PF)$.

Proof. The utility set U is a subset of a polyhedron with constraints $0 \le u_i \le L_i, i \in N$. Let K denote a transformation of U such that $K = \{(k_1, k_2, \dots, k_n) | (k_1 L_1, k_2 L_2, \dots, k_n L_n) \in U\}$. Then K is a subset of the unit cubic, i.e., $K \subseteq [0,1]^n$.

By Proposition 1, there exist $a_i \in [1/n, 1]$, $i \in N$, such that $K \subseteq K' = \{k := (k_1, k_2, \dots, k_n) \mid \sum_{i=1}^n a_i k_i \le 1, 0 \le k_i \le 1, i \in N\}$, and $\max\{\sum_{i=1}^n \log(k_i) | k \in K\} = \max\{\sum_{i=1}^n \log(k_i) | k \in K'\}$.

Finally, we derive the expression of U' corresponding to K'. Let $c_i = a_i / L_i \in [1/nL_i, 1/L_i]$, $i \in N$. That is, $U' = \{u | (u_1/L_1, u_2/L_2, \dots, u_n/L_n) \in K'\} = \{u | \sum_{i=1}^n c_i u_i \le 1, 0 \le u_i \le L_i, i \in N\}$. Then, $\max\{\sum_{i=1}^n \log(u_i) | u \in U\} = \max\{\sum_{i=1}^n \log(k_i) | k \in K\} + \sum_{i=1}^n \log(L_i) = \max\{\sum_{i=1}^n \log(k_i) | k \in K'\} + \sum_{i=1}^n \log(L_i) = \max\{\sum_{i=1}^n \log(u_i) | u \in U'\}$.

■

3.2.1. Proportional Fairness To find the PF solution in the unequal maximum achievable utilities case we solve the following optimization Problem (12):

$$\max_{(u_1, u_2, \cdots, u_n)} \quad \sum_{i=1}^n log(u_i), \tag{12a}$$
$$\text{s.t.} \quad 0 \le u_i \le L_i, \qquad \forall i \in N, \tag{12b}$$
$$\sum_{i=1}^n c_i u_i \le 1. \tag{12c}$$

Let $k_i = u_i / L_i$, then Problem (12) can be written as Problem (13):

$$\max_{(k_1, k_2, \cdots, k_n)} \quad \sum_{i=1}^n log(k_i) + \sum_{i=1}^n log(L_i), \tag{13a}$$
$$\text{s.t.} \quad 0 \le k_i \le 1, \qquad \forall i \in N, \tag{13b}$$
$$\sum_{i=1}^n c_i L_i k_i \le 1. \tag{13c}$$

The PF solution in the unequal maximum achievable utilities case is provided in the next proposition. Its proof is essentially identical to the proof of Proposition 2 and is therefore omitted.

Proposition 5. *The unique optimal solutions to Problems* (12) *and* (13) *are* $k^{PF} = (1/(nc_1 L_1), 1/(nc_2 L_2), \dots, 1/(nc_n L_n))$ *and* $u^{PF} = (1/(nc_1), 1/(nc_2), \dots, 1/(nc_n))$, *respectively.*

3.2.2. Utilitarian Solution By Proposition 4, the utilitarian solution in the unequal maximum achievable utilities case is the optimal solution to

$$\max_{(u_1,u_2,\cdots,u_n)} \quad \sum_{i=1}^{n} u_i, \tag{14a}$$
$$\text{s.t.} \quad 0 \le u_i \le L_i, \qquad \forall i \in N, \tag{14b}$$
$$\sum_{i=1}^{n} c_i u_i \le 1. \tag{14c}$$

Let $k_i = u_i/L_i$, then the above optimization problem can be written as Problem (15):

$$\max_{(k_1,k_2,\cdots,k_n)} \quad \sum_{i=1}^{n} L_i k_i, \tag{15a}$$
$$\text{s.t.} \quad 0 \le k_i \le 1, \qquad \forall i \in N, \tag{15b}$$
$$\sum_{i=1}^{n} c_i L_i k_i \le 1. \tag{15c}$$

Problem (15) is a linear relaxation of the 0-1 knapsack problem with rewards L_i and costs $c_i L_i$. Thus, its optimal solution is

$$k_{\sigma(i)} = \begin{cases} 1 & i = 1, \ldots, l(c) \\ \delta(c) & i = l(c)+1 \\ 0 & i = l(c)+2, \ldots, n, \end{cases}$$

where

$$l(c) := \max\{j \mid \sum_{i=1}^{j} c_{\sigma(i)} L_{\sigma(i)} \le 1\}, \qquad \delta(c) := \frac{1 - \sum_{i=1}^{l(c)} c_{\sigma(i)} L_{\sigma(i)}}{c_{\sigma(l(c)+1)} L_{\sigma(l(c)+1)}}, \tag{16}$$

and $\sigma(i)$ is the index of the i^{th} smallest element in vector c.

Without loss of generality, we assume that the L_i's are decreasing, i.e. $L_1 \ge L_2 \ge \cdots \ge L_n$. Then let us consider the following Problem (17),

$$\max_{(k_1,k_2,\cdots,k_n)} \quad \sum_{i=1}^{n} L_i k_i, \tag{17a}$$
$$\text{s.t.} \quad 0 \le k_i \le 1, \qquad i \in N, \tag{17b}$$
$$\sum_{i=1}^{n} c_{\sigma(i)} L_i k_i \le 1. \tag{17c}$$

Problem (17) is a knapsack problem with rewards L_i and costs $c_{\sigma(i)} L_i$. Compared to problem (15), in Problem (17), items with larger rewards, L_i, have smaller costs, $c_{\sigma(i)} L_i$. Thus, the optimal value of Problem (17) exceeds that on Problem (15). Note further that replacing all c_i with $c_{\sigma(i)}$ does not affect the total sum of the players' utilities in the PF solution. Thus, optimality would be achieved when $c_1 \le c_2 \le \cdots \le c_n$, and we can therefore assume in the sequel that $c_1 \le c_2 \le \cdots \le c_n$.

3.2.3. The Price of Proportional Fairness Similar to our analysis of the equal maximum achievable utilities case, to calculate the upper bound of the price of proportional fairness, we solve the following optimization Problem (18):

$$\min_{c} \quad \left(\frac{\sum_{i=1}^{n} u_i^{PF}}{\sum_{i=1}^{n} u_i^{*}} =\right) \frac{\sum_{i=1}^{n} \frac{1}{nc_i}}{\sum_{i=1}^{l} L_i + \frac{1-\sum_{i=1}^{l} c_i L_i}{c_{l+1}}}, \tag{18a}$$

$$\text{s.t.} \quad \frac{1}{nL_1} \le c_1 \le c_2 \le \cdots \le c_n \le \frac{1}{L_n}, \tag{18b}$$

$$\frac{1}{n} \le c_i L_i \le 1, \qquad i \in N, \tag{18c}$$

$$\sum_{i=1}^{l} c_i L_i \le 1, \tag{18d}$$

$$\sum_{i=1}^{l+1} c_i L_i > 1. \tag{18e}$$

Let $c^* := (c_1^*, c_2^*, \ldots, c_n^*)$ denote an optimal solution for (18). By characterizing c^*, we can derive a tight bound of the POF for the proportional fairness case when players have unequal maximum achievable utilities. Let l denote $l(c)$. Then, Constraints (18d) and (18e) ensure that l satisfies the definition of $l(c)$, given in Expression (16).

Theorem 2. *Consider a resource allocation problem with n players, $n \ge 2$, defined over a utility set U which is assumed to be compact, convex and monotone. Denote by L_i the maximum achievable utility of player i, and assume, without loss of generality, that $L_1 \ge L_2 \ge \cdots \ge L_n$. Then, the upper bound of the price of proportional fairness, $POF(U;PF)$, is:*

(1) If $\sum_{i=1}^{n} L_i \cdot L_2 \le L_1^2$,

$$POF(U;PF) \le 1 - \min_{l \in \mathbb{N}} \frac{(\sum_{i=1}^{l} \sqrt{L_i})^2 + \sum_{i=l+1}^{n} L_i}{n \sum_{i=1}^{l} L_i};$$

(2) If $\sum_{i=1}^{n} L_i \cdot L_2 > L_1^2$,

$$POF(U;PF) \le 1 - \min \left\{ \min_{l \in \mathbb{N}} \frac{(\sum_{i=1}^{l} \sqrt{L_i})^2 + \sum_{i=l+1}^{n} L_i}{n \sum_{i=1}^{l} L_i}, \frac{(\sqrt{L_2^2 + 2L_1L_2 + (L_1+L_2)\sum_{i=3}^{n} L_i} + \sqrt{L_1L_2})^2}{n(L_1+L_2)^2} \right\}.$$

Moreover, for any given L, the above bounds are tight in the sense that there exists utility set U for which equality hold in the above inequalities.

Proof.

We first characterize the optimal solution, c^*, to Problem (18), which is using the following claims, whose proofs are provided in Appendix EC.3. Let $l := l(c^*)$ and $x := c_{l+1}$.

CLAIM 7. *$c_i^* = 1/L_i$, for $i = l+2, \cdots, n$.*

CLAIM 8. *$c_1^*\sqrt{L_1} = c_2^*\sqrt{L_2} = \cdots = c_l^*\sqrt{L_l}$.*

As a result of Claim 8, we may denote $y := c_i\sqrt{L_i}$ for $i = 1, 2, \cdots, l$. Then, Problem (18) can be written as follow,

$$
\begin{aligned}
\min_{x,y,l} \quad & \frac{\frac{1}{y}\sum_{i=1}^{l}\sqrt{L_i} + \frac{1}{x} + \sum_{i=l+2}^{n} L_i}{\sum_{i=1}^{l} L_i + \frac{1-y\sum_{i=1}^{l}\sqrt{L_i}}{x}}, && \text{(19a)}\\
\text{s.t.} \quad & \frac{y}{\sqrt{L_l}} \le x \le \frac{1}{L_{l+1}}, && \text{(19b)}\\
& x \ge \frac{1}{nL_{l+1}}, && \text{(19c)}\\
& \frac{1}{n\sqrt{L_l}} \le y \le \frac{1}{\sum_{i=1}^{l}\sqrt{L_i}}, && \text{(19d)}\\
& y\sum_{i=1}^{l}\sqrt{L_i} + xL_{l+1} > 1. && \text{(19e)}
\end{aligned}
$$

For convenience, we let $A(l)$ denote $\sum_{i=1}^{l}\sqrt{L_i}$, $B(l)$ denote $\sum_{i=l+2}^{n} L_i$, and $M(l)$ denote $\sum_{i=1}^{l} L_i$. Problem (20) is a relaxation of Problem (19), derived therefrom by removing the left-hand-side constraints of (19b) and (19d).

$$
\begin{aligned}
\min_{x,y,l} \quad & f_2(x,y,l) := \frac{\frac{1}{y}A(l) + \frac{1}{x} + B(l)}{M(l) + \frac{1-yA(l)}{x}}, && \text{(20a)}\\
\text{s.t.} \quad & \frac{1}{nL_{l+1}} \le x \le \frac{1}{L_{l+1}}, && \text{(20b)}\\
& y \le \frac{1}{A(l)}, && \text{(20c)}\\
& yA(l) + xL_{l+1} > 1. && \text{(20d)}
\end{aligned}
$$

We next characterize an optimal solution, (x^*, y^*, l^*), to Problem (20). We start by characterizing x^* and y^* in the next two claims. The proofs of these claims, as well as the proofs of the other claims used to prove Theorem 2 are provided in Appendix EC.3.

CLAIM 9. *$x^* = 1/L_{l^*+1}$.*

CLAIM 10. *(1) When $l^* = 1$ with $(B(0)+L_1)L_2 \le L_1^2$, or when $l^* \ge 2$, $y^* = 1/A(l^*)$.*
(2) When $l^* = 1$ with $(B(0)+L_1)L_2 > L_1^2$, $y^* = \tilde{y} := (-\sqrt{L_1} + \sqrt{B(0)+L_1+B(0)L_1/L_2})/B(0)$.

Finally, we are left to characterize l^*. If $(B(0)+L_1)L_2 \le L_1^2$, then, by Claim 9 and 10 the objective function attains the minimum at $x^* = 1/L_{l^*+1}, y^* = 1/A(l^*)$, regardless of whether $l^* = 1$ or $l^* \ge 2$. Let

$$h(l) := f_2\left(\frac{1}{L_{l+1}}, \frac{1}{A(l)}, l\right) = \frac{(A(l))^2 + L_{l+1} + B(l)}{M(l)} = \frac{(A(l))^2 + B(l-1)}{M(l)}.$$

In Claim 11 below we prove that $h(l)$ is decreasing for $l \in [1, l^*]$ and increasing for $l \in [l^*, n-1]$. Thus, we can find l^* by comparing $h(l)$ for successive values of l. If $(B(0)+L_1)L_2 > L_1^2$, then $x = 1/L_2$, $y = \tilde{y}$, $l = 1$, as well as $x = 1/L_{l+1}$, $y = 1/A(l)$, $l = 2, 3, \ldots, n-1$, are all candidates for an optimal solution. So we need to compare the values of $f_2(1/L_2, \tilde{y}, 1)$ with $h(l)$, $l = 2, \cdots, n-1$ as well. Such comparison admits a concise characterization as proved in the following claim.

Claim 11. *$h(l)$ is decreasing when $l \in [1, l^*]$, increasing when $l \in [l^*, n-1]$, and $l^* = \arg\min_l h(l)$ satisfies the following two inequalities if $l^* \in \{2, 3, \cdots, n-2\}$:*

$$\begin{cases} \sqrt{L_{l^*+1}}[(A(l^*))^2 + B(l^*-1)] \le 2A(l^*)M(l^*), & (21) \\ \sqrt{L_{l^*}}[(A(l^*))^2 + B(l^*-1)] \ge 2M(l^*)A(l^*-1), & (22) \end{cases}$$

where if $l^* = 1$, only (21) is satisfied, and if $l^* = n-1$, only (22) is satisfied.

So far we have characterized an optimal solution (x^*, y^*, l^*) to (20). However, since (20) is a relaxation of the original Problem (19), we have to show that (x^*, y^*, l^*) is feasible to the original optimization Problem (19). It would then follow that (x^*, y^*, l^*) is also an optimal solution to (19). By explicitly constructing a solution to (19), we have also proven the tightness of the upper bound for the POF.

Claim 12. *(x^*, y^*, l^*), as characterized in Claims 7-9, is feasible and thus optimal to Problem (19).*

■

Theorem 2 reduces Theorem 1 when players have equal maximum achievable utilities. To see that, let $L_i \equiv 1$, for $i \in N$. In this case, $\sum_{i=1}^n L_i \cdot L_2 - L_1^2 = n - 1 > 0$. Then, by Theorem 2, we have

$$\begin{aligned} POF(U;PF) &\le 1 - \min\left\{ \min_{l\in\mathbb{N}} \frac{(\sum_{i=1}^l \sqrt{L_i})^2 + \sum_{i=l+1}^n L_i}{n\sum_{i=1}^l L_i}, \right. \\ &\qquad \left. \frac{(\sqrt{L_2^2 + 2L_1L_2 + (L_1+L_2)\sum_{i=3}^n L_i} + \sqrt{L_1L_2})^2}{n(L_1+L_2)^2} \right\} \\ &= 1 - \frac{1}{n}\min\left\{ \min_{l\in\mathbb{N}} \frac{l^2+n-l}{l}, \frac{\sqrt{2n-1}+n}{2} \right\} \\ &\le 1 - \min_{l\in\mathbb{N}} \frac{1}{n}(l + \frac{n}{l} - 1), \end{aligned} \tag{23}$$

where the last inequality follows from Lemma 2, whose proof is provided in Appendix EC.4.

Lemma 2. *For $n \ge 3$,*

$$\min_{l\in\mathbb{N}} \frac{l^2+n-l}{l} \le \frac{\sqrt{2n-1}+n}{2}. \tag{24}$$

Since the negative term in the last line of (23) coincides with the right hand side term in (11), we conclude that the upper bound for proportional POF when the maximum achievable utilities by the players are not necessarily equal, given by Theorem 2, reduces to the bound given by Theorem 1 when the maximum achievable utilities of all players are equal.

3.2.4. The worst case of maximum achievable utilities Theorem 2 provides a tight upper bound for the price of PF for a given set of maximum achievable utilities by the players. We next provide an upper bound for the price of PF when the maximum achievable utilities for any of the players can assume any positive finite value in (0,1].

Let $UB(L;n,PF)$ denote an upper bound for the price of PF for a fixed number of players, n, and for a given maximum achievable utility vector L.

PROPOSITION 6. *Let $L_i \in (0,1]$ denote the maximum achievable utility of player i, and assume, without loss of generality, that $L_1 \geq L_2 \geq \cdots \geq L_n$. Then,*

$$\sup_{L: L_i \in (0,1], L_1 \geq \ldots \geq L_n} UB(L;n,PF) = 1 - \frac{1}{n}.$$

Proof.

From Theorem 2, $UB(L;n,PF)$ has the following two possible expressions:

$$UB(L;n,PF) = 1 - \min_{l \in \mathbb{N}} \frac{(\sum_{i=1}^{l} \sqrt{L_i})^2 + \sum_{i=l+1}^{n} L_i}{n \sum_{i=1}^{l} L_i}; \tag{25a}$$

$$UB(L;n,PF) = 1 - \frac{(\sqrt{L_2^2 + 2L_1L_2 + (L_1+L_2)\sum_{i=3}^{n} L_i} + \sqrt{L_1L_2})^2}{n(L_1+L_2)^2} \text{(Only possible when } \sum_{i=1}^{n} L_i \cdot L_2 > L_1^2). \tag{25b}$$

(1) If $UB(L;n,PF)$ has the Expression (25a), then

$$\begin{aligned} UB(L;n,PF) &= 1 - \min_{l \in \mathbb{N}} \frac{(\sum_{i=1}^{l} \sqrt{L_i})^2 + \sum_{i=l+1}^{n} L_i}{n \sum_{i=1}^{l} L_i} \leq 1 - \min_{l \in \mathbb{N}} \frac{\sum_{i=1}^{l} L_i + \sum_{i=l+1}^{n} L_i}{n \sum_{i=1}^{l} L_i} \\ &= 1 - \frac{1}{n} - \min_{l \in \mathbb{N}} \frac{\sum_{i=l+1}^{n} L_i}{n \sum_{i=1}^{l} L_i} < 1 - \frac{1}{n}. \end{aligned} \tag{26}$$

(2) If $UB(L;n,PF)$ has the Expression (25b), then

$$\begin{aligned} UB(L;n,PF) &= 1 - \frac{(\sqrt{L_2^2 + 2L_1L_2 + (L_1+L_2)\sum_{i=3}^n L_i} + \sqrt{L_1L_2})^2}{n(L_1+L_2)^2} \\ &= 1 - \frac{(\sqrt{(L_1+L_2)\sum_{i=1}^n L_i - L_1^2} + \sqrt{L_1L_2})^2}{n(L_1+L_2)^2} \\ &< 1 - \frac{(\sqrt{(L_1+L_2)\frac{L_1^2}{L_2} - L_1^2} + \sqrt{L_1L_2})^2}{n(L_1+L_2)^2} \qquad (\text{By } \sum_{i=1}^n L_i \cdot L_2 > L_1^2) \qquad (27) \\ &= 1 - \frac{(L_1\sqrt{\frac{L_1}{L_2}} + \sqrt{L_1L_2})^2}{n(L_1+L_2)^2} \\ &< 1 - \frac{(L_1+L_2)^2}{n(L_1+L_2)^2} = 1 - \frac{1}{n}. \end{aligned}$$

Thus, for any vector L of maximum achievable utilities, $UB(L;n,PF) < 1 - 1/n$. We next prove that $\sup UB(L;n,PF) = 1 - 1/n$.

Indeed, we will show that for any positive δ and a number smaller than $1 - 1/n - \delta$, there exists a vector of maximum achievable utilities L^*, for which $UB(L^*;n,PF) > 1 - 1/n - \delta$. Specifically, let $L_1^* = 1$, and let $L_i^* = \epsilon, i = 2,3...,n$, such that ϵ is small so that $\sum_{i=1}^n L_i^* \cdot L_2^* \le L_1^{*2}$. Clearly, for ϵ small enough, i.e., $\epsilon \le (\sqrt{4n-3} - 1)/(2(n-1))$, this inequality is satisfied. Then, let $\epsilon = \min\{n\delta/(2(n-1)), (\sqrt{4n-3} - 1)/(2(n-1))\}$,

$$\begin{aligned} UB(L^*;n,PF) &= 1 - \min_{l\in\mathbb{N}} \frac{(\sqrt{L} + (l-1)\sqrt{\epsilon})^2 + (n-l)\epsilon}{n(L+(l-1)\epsilon)} \\ &\ge 1 - \frac{L+(n-1)\epsilon}{nL} = 1 - \frac{1}{n} - \frac{\delta}{2} > 1 - \frac{1}{n} - \delta. \end{aligned}$$

■

Remark 1. We note that one can also easily show that for a fixed number of players, n, the supremum of the BFT bound for the price of proportional fairness over all maximum achievable utilities is also $1 - 1/n$.

3.2.5. Sensitivity of the bounds to the variability of the maximum achievable utilities

We next investigate the sensitivity of the bounds for the price of PF to the variability of the maximum achievable utilities, $L = (L_1, \ldots, L_n)$. Figure 3 displays the case of $n = 2$, wherein the maximum attainable utility of player 1, L_1, is set at 1, and L_2 varies within the interval $(0,1]$. When $L_2 = 1$, the sample variance is 0, corresponding to the case of equal maximum achievable utilities. In this case, the BFT bound is 8.6%, and our bound is 6.7%, consistent with the results presented in BFT (2011) and Theorem 1, respectively. As L_2 approaches 0, the sample variance approaches its maximum value of 0.25, at which point both bounds converge to 0.5, which is consistent with Proposition 6 and Remark 1. Figure 4 also clearly illustrates that the improvement of our bound over the BFT

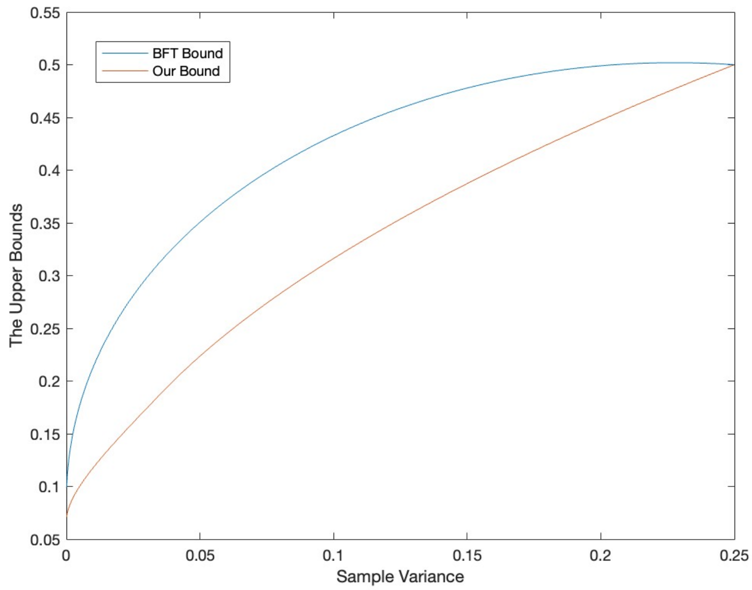


Figure 3 The two upper bounds of the price of proportional fairness as a function of sample variance for case of $n = 2$.

bound for the price of PF is relatively substantial when the maximum achievable utilities have an intermediate sample variance.

We have also compared the sensitivity of the two bounds to the variability of the maximum achievable utilities for the case with $n = 9$ players. We have generated 100 groups of players' maximum achievable utilities $L = (L_1, \ldots, L_n)$. In the t^{th} group, each element of L is generated from an independent truncated normal distribution with mean 1 and standard deviation $\sigma_t = 0.01(t-1)$, $t = 1, \ldots, 100$. The normal distribution was truncated to ensure that all elements of L are positive. The two bounds are displayed in Figure 4 as a function of the sample variance. When the sample variance is zero, both bounds coincide, since when $\sqrt{n}$ is integer the BFT bound is tight when the maximum achievable utilities are equal. But as Figure 4 clearly demonstrates, the quality of the BFT bound deteriorates at a higher rate than our bound when the variability of the maximum achievable utilities increases. Indeed, the difference between the two bounds in the unequal maximum achievable utilities case is more significant than in the case of equal maximum achievable utilities, as it was displayed in Figure 1.

4. Upper Bounds for The Price of Max-Min Fairness

BFT (2011) have derived a tight bound for the price of max-min fairness when the maximum achievable utilities by all players are equal. In this section we improve upon the BFT bound for the price of max-min fairness when the players maximum achievable utilities are not necessarily equal. We prove

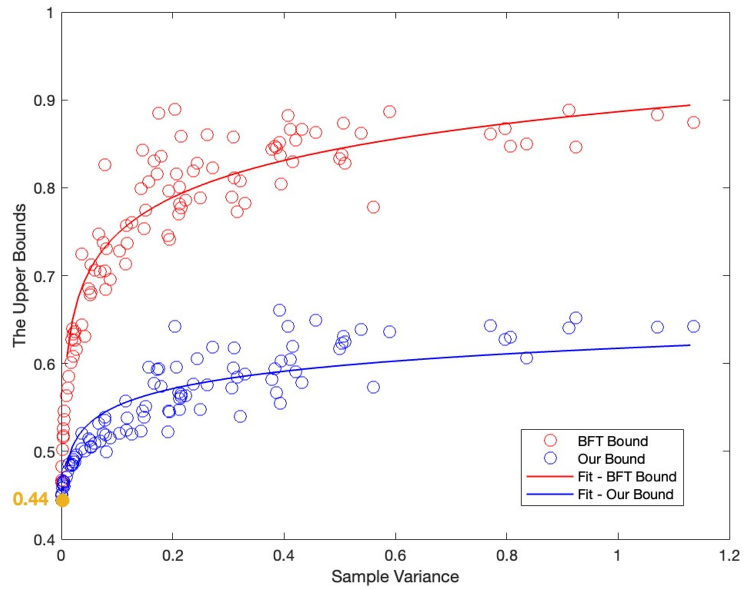


Figure 4 **The two upper bounds of the price of proportional fairness as a function of sample variance.**

that our bound is tight and investigate the difference between our bound and the BFT bound for the unequal case.

4.1. The Max-Min Fairness Solution

Similar to the PF case, we first utilize a super set U' of the utility set U, to derive an upper bound for the price of MMF in a tractable form.

Proposition 7. *Suppose* $U \subseteq \{u \,|\, 0 \le u_i \le L_i, i \in N\}$ *is a convex, compact, and monotone utility set, and* $\max\{u_i \,|\, u \in U\} = L_i$ *for* $i \in N$*. Then there exist* $c_i \in \mathbb{R}$ *for* $i \in N$*, such that* $U \subseteq U' := \{u \,|\, \sum_{i=1}^n c_i u_i \le 1, 0 \le u_i \le L_i,\ i \in N\}$*, and* $\max\{\min_{i\in N}(u_i/L_i) | u \in U\} = \max\{\min_{i\in N}(u_i/L_i) | u \in U'\}$*. Consequently,*

$$POF(U; MMF) \le 1 - \frac{\sum_{i=1}^n L_i}{\sum_{i=1}^n c_i L_i \sum_{i=1}^n u^*(U')_i}.$$

Proof. The MMF solution first maximizes the minimum among the ratios of the players' utilities to their respective maximum achievable utilities, and so on. We define k_i as the ratio of player i's utility to their maximum achievable utility, i.e., $k_i := u_i^{MMF}(U)/L_i$, then $u^{MMF}(U) = (k_1 L_1, \dots, k_n L_n)$.

Let ϕ denote the maximum ratio of a player's utility to their respective maximum achievable utility that all players can derive simultaneously. Then $k_i \ge \phi$ for all $i \in N$. Since every player can achieve in U their maximum achievable utility, $(0, \dots, L_i, \dots, 0) \in U$ for all $i \in N$. Then, by convexity, $(L_1/n, \dots, L_n/n) \in U$, so ϕ is at least $1/n$. Further, since players cannot achieve a utility that exceed their respective maximum achievable utility, $\phi \in [1/n, 1]$.

Then, $u^{MMF}(U) \geq (\phi L_1, \ldots, \phi L_n)$, and we have

$$POF(U; MMF) = 1 - \frac{\sum_{i=1}^{n} u_i^{MMF}(U)}{\sum_{i=1}^{n} u_i^*(U)} \leq 1 - \frac{\phi \sum_{i=1}^{n} L_i}{\sum_{i=1}^{n} u_i^*(U)}.$$

Since $\overline{u} := (\phi L_1, \ldots, \phi L_n)$ is a boundary point in U, by the supporting hyperplane theorem, there exists a vector $v \neq 0$, such that

$$v^T \overline{u} \leq v^T u, \forall u \in U. \tag{28}$$

Since $0 \in U$, we have $v^T \overline{u} \leq 0$. We claim that $v^T \overline{u} < 0$. Indeed, suppose, on the contrary, that $v^T \overline{u} = 0$, i.e., $\sum_{i=1}^{n} v_i L_i = 0$. Then, since $(0, \ldots, L_i, \ldots, 0) \in U$ for all i, by (28), we have $v_i L_i \geq 0$ for all i, which implies that $v_i L_i = 0$ for all i. However, since $L_i > 0$, we must have that $v_i = 0$ for all i, contradicting the fact that $v \neq 0$.

Let $c_i := v_i / \sum_{i=1}^{n} v_i \overline{u}_i = v_i / \sum_{i=1}^{n} v_i L_i \phi$. Then, the supporting hyperplane can be written as $\sum_{i=1}^{n} c_i u_i \leq 1$ for all $u \in U$, and $\sum_{i=1}^{n} c_i L_i = 1/\phi$, implies that $\phi = 1/\sum_{i=1}^{n} c_i L_i$.

Let $U' = \{(u_1, u_2, \ldots, u_n) \mid \sum_{i=1}^{n} c_i u_i \leq 1,\ 0 \leq u_i \leq L_i,\ i = 1, \ldots, n\}$. Then, $U \subseteq U'$, $\sum_{i=1}^{n} u_i^*(U) \leq \sum_{i=1}^{n} u_i^*(U')$, and

$$POF(U; MMF) \leq 1 - \frac{\phi \sum_{i=1}^{n} L_i}{\sum_{i=1}^{n} u_i^*(U)} \leq 1 - \frac{\phi \sum_{i=1}^{n} L_i}{\sum_{i=1}^{n} u_i^*(U')} = 1 - \frac{\sum_{i=1}^{n} L_i}{\sum_{i=1}^{n} c_i L_i \sum_{i=1}^{n} u_i^*(U')}. \tag{29}$$

■

4.2. The Utilitarian Solution

The approach to find the utilitarian solution for U' is similar to that used to derive the price of proportional fairness, which involves solving the following optimization Problem (30):

$$\max_{(u_1, u_2, \cdots, u_n)} \quad \sum_{i=1}^{n} u_i, \tag{30a}$$

$$\text{s.t.} \quad 0 \leq u_i \leq L_i, \qquad \forall i \in N, \tag{30b}$$

$$\sum_{i=1}^{n} c_i u_i \leq 1. \tag{30c}$$

Let $k_i = u_i / L_i$ for all i. Then Problem (30) can be formulated as Problem (31):

$$\max_{(k_1, k_2, \cdots, k_n)} \quad \sum_{i=1}^{n} L_i k_i, \tag{31a}$$

$$\text{s.t.} \quad 0 \leq k_i \leq 1, \qquad \forall i \in N, \tag{31b}$$

$$\sum_{i=1}^{n} c_i L_i k_i \leq 1. \tag{31c}$$

Problem (31) is a linear relaxation of the knapsack problem with rewards L_i and costs c_iL_i. Without loss of generality, we assume that c_i is increasing in i, i.e.,

$$c_1 \le c_2 \le \cdots \le c_n. \tag{32}$$

As before, we define

$$l(c) := \max\{j \mid \sum_{i=1}^{j} c_iL_i \le 1\}, \qquad \delta(c) := \frac{1 - \sum_{i=1}^{l(c)} c_iL_i}{c_{l(c)+1}L_{l(c)+1}} \in [0,1). \tag{33}$$

Then the optimal solution to Problem (31) is given by

$$k_i = \begin{cases} 1 & i = 1, \dots, l(c) \\ \delta(c) & i = l(c)+1 \\ 0 & i = l(c)+2, \dots, n, \end{cases}$$

and the optimal value is $\sum_{i=1}^{l(c)} L_i + \delta(c)L_{l(c)+1}$.

4.3. The Price of MMF

In order to calculate the upper bound of the price of max-min fairness, it suffices to minimize the negative term in the last expression in (29), by incorporating the optimal solution to Problem (31) derived above. Let l denote $l(c)$, and consider the optimization Problem (34):

$$\min_{c,l} \quad (f_3(c,l;L) :=) \, \frac{c_{l+1}\sum_{i=1}^{n} L_i}{\left(\sum_{i=1}^{n} c_iL_i\right) \cdot \left(c_{l+1}\sum_{i=1}^{l} L_i + 1 - \sum_{i=1}^{l} c_iL_i\right)}, \tag{34a}$$

$$\text{s.t.} \quad c_1 \le c_2 \le \cdots \le c_n, \tag{34b}$$

$$c_iL_i \le 1, \qquad i \in N, \tag{34c}$$

$$\sum_{i=1}^{l} c_iL_i \le 1, \tag{34d}$$

$$\sum_{i=1}^{l+1} c_iL_i > 1. \tag{34e}$$

The first Constraint, (34b), is from (32). Since $(0, \dots, L_i, \dots, 0) \in U$ for all i and $U \subseteq U'$, we have $(0, \dots, L_i, \dots, 0) \in U'$, which implies Constraint (34c). Constraints (34d) and (34e) ensure that l satisfies the definition of $l(c)$, given in Expression (33).

Theorem 3. *Consider a resource allocation problem with n players, where $n \ge 2$, and assume the utility set U is compact and convex. Let L_i denote the maximum achievable utility of player i, and assume, without loss of generality, that $L_1 \ge L_2 \ge \cdots \ge L_n$. Let $S_1 := \{l \in \mathbb{N}^+ : (n-l-1)L_{l+1} <$*

$\sum_{i=1}^{l} L_i \leq (n-l+1)L_{l+1}\}$, *and let* $S_2 := \{l \in \mathbb{N}^+ : \sum_{i=1}^{l} L_i > (n-l+1)L_{l+1}\}$. *Then, the upper bound of the price of MMF is:*

(1) If $S_1 \neq \emptyset$,

$$POF(U; MMF) \leq 1 - \frac{4L_{l^*+1}\sum_{i=1}^{n} L_i}{(\sum_{i=1}^{l^*} L_i + (n-l^*+1)L_{l^*+1})^2}, \qquad \textit{where } l^* = \max\{S_1\}; \tag{35}$$

(2) If $S_1 = \emptyset$,

$$POF(U; MMF) \leq 1 - \frac{\sum_{i=1}^{n} L_i}{\sum_{i=1}^{l^*} L_i(n-l^*+1)}, \qquad \textit{where } l^* = \min\{S_2\}, \tag{36}$$

where $S_1 := \{l \in \mathbb{N}^+ : (n-l-1)L_{l+1} < \sum_{i=1}^{l} L_i \leq (n-l+1)L_{l+1}\}$, $S_2 := \{l \in \mathbb{N}^+ : \sum_{i=1}^{l} L_i > (n-l+1)L_{l+1}\}$.

Moreover, the above bounds are tight in the sense that there exists a utility set U for which equality holds in the above inequalities.

Proof.

The proofs of all claims used to prove Theorem 3 are in Appendix EC.5.

Let $c^* := (c_i^*,\ i = 1, 2, \cdots, n)$ denote the optimal solution of Problem (34). As before, we first characterize c_i^* for $i \geq l+2$.

Claim 13. $c_i^* = 1/L_i$ *for* $i = l+2, \cdots, n$.

To simplify the notation, we define

$$x := c_{l+1}, y := \sum_{i=1}^{l} c_i L_i, \text{ and } A := \sum_{i=1}^{l} L_i.$$

According to Claim 13, Problem (34) can be reformulated as Problem (37):

$$\begin{aligned}
\min_{y,x,l} \quad & \frac{x\sum_{i=1}^{n} L_i}{(y+n-l-1+xL_{l+1})(Ax+1-y)}, && (37a)\\
\text{s.t.} \quad & y \leq Ax, && (37b)\\
& 0 \leq xL_{l+1} \leq 1, && (37c)\\
& 0 \leq y \leq 1, && (37d)\\
& y + xL_{l+1} > 1. && (37e)
\end{aligned}$$

In Problem (37), we will first fix the value of $l, l \in [1, n-1]$ and characterize the optimal values of y, x. Note that if $l = n$, then $\sum_{i=1}^{n} c_i L_i \leq 1$ and thus $L := (L_1, L_2, \ldots, L_n) \in U$. In this case, both the utilitarian solution and the MMF solution are L, with a corresponding POF $= 0$, which is not an interesting case to explore.

Let $g(x,y;l)$ denote the objective function without the constant $\sum_{i=1}^{n} L_i$ in (37a), i.e.,

$$g(x,y;l) := \frac{x}{(y+n-l-1+xL_{l+1})(Ax+1-y)}.$$

For a given l, we have the following characterization of x^*, y^*, and $g^*(x^*,y^*;l)$.

Claim 14. *If $A \le (n-l-1)L_{l+1}$,*

$$x^* \in [\frac{1}{A+L_{l+1}}, \frac{1}{L_{l+1}}], \quad y^* = 1 - x^* L_{l+1}, \quad g(x^*,y^*;l) = \frac{1}{(n-l)(A+L_{l+1})};$$

If $(n-l-1)L_{l+1} < A \le (n-l+1)L_{l+1}$,

$$x^* = \frac{1}{L_{l+1}}, \quad y^* = \frac{1}{2}(\frac{A}{L_{l+1}} - n + l + 1), \quad g(x^*,y^*;l) = \frac{4L_{l+1}}{(A+(n-l+1)L_{l+1})^2};$$

If $A > (n-l+1)L_{l+1}$,

$$x^* = \frac{1}{L_{l+1}}, \quad y^* = 1, \quad g(x^*,y^*;l) = \frac{1}{A(n-l+1)}.$$

We next characterize the order of the $L_i's$ at optimality. For convenience, we introduce $p(A, L_{l+1}; l)$, defined as follows:

$$p(A, L_{l+1}; l)(:= g(x^*,y^*,l)) = \begin{cases} \frac{1}{(n-l)(A+L_{l+1})}, & A \le (n-l-1)L_{l+1}, \\ \frac{4L_{l+1}}{(A+(n-l+1)L_{l+1})^2}, & (n-l-1)L_{l+1} < A \le (n-l+1)L_{l+1}, \\ \frac{1}{A(n-l+1)}, & A > (n-l+1)L_{l+1}. \end{cases} \tag{38}$$

Claim 15. *$p(A, L_{l+1}; l)$ is decreasing both in A and L_{l+1}.*

Since A is the sum of L_1 to L_l, the internal order of L_1 to L_l does not affect the value of A and thus, does not affect the objective function value. Therefore, without loss of generality, we assume that L_1 to L_l are arranged in a descending order. Then, according to Claim 15, the permutation of the $L_i's$ that would minimize $p(A, L_{l+1}; l)$ should satisfy: $L_1 \ge L_2 \ge \cdots \ge L_l \ge L_{l+2}$, $L_{l+1} \ge L_{l+2}$ and $L_{l+2} \ge L_{l+3} \ge \cdots \ge L_n$. In fact, in Claim 16 below we prove that the $L_i's$ should be in descending order overall, i.e., $L_1 \ge L_2 \ge \cdots \ge L_n$.

Claim 16. *The value of $p(A, L_{l+1}; l)$ is minimized with respect to the $L_i, i = 1, \ldots, n$, if they are arranged in a descending order.*

Next we discuss the value of l at optimality. For convenience, we define

$$S_0 := \{l \in \mathbb{N}^+ : \sum_{i=1}^{l} L_i \le (n-l-1)L_{l+1}\},$$

$$S_1 := \{l \in \mathbb{N}^+ : (n-l-1)L_{l+1} < \sum_{i=1}^{l} L_i \le (n-l+1)L_{l+1}\},$$

$$S_2 := \{l \in \mathbb{N}^+ : \sum_{i=1}^{l} L_i > (n-l+1)L_{l+1}\}.$$

Claim 17. *Let l^* denote an optimal value of l. If $S_1 \neq \emptyset$, $l^* = \max\{S_1\}$, otherwise $l^* = \min\{S_2\}$.*

Now, $l^* = \max\{S_1\}$, corresponds to the second case in Claim 14. Thus, $g(x^*, y^*; l^*) = 4L_{l^*+1}/(A + (n - l^* + 1)L_{l^*+1})^2$ and $1 - POF(U; MMF)$, which is the objective function of (37), is given by $4L_{l^*+1} \sum_{i=1}^n L_i/(A + (n - l^* + 1)L_{l^*+1})^2$. Similarly, $l^* = \min\{S_2\}$ corresponds to the third case of Claim 14. Thus, $g(x^*, y^*; l^*) = 1/A(n - l^* + 1)$ and $1 - POF(U; MMF) = \sum_{i=1}^n L_i/A(n - l^* + 1)$, and we have derived the expressions for the upper bound for the price of MMF given in Theorem 3.

Finally, we provide an instance of U to illustrate that the above valuations of l^*, x^*, and y^* (or c^*) can be achieved, which implies that the upper bound in Theorem 3 is tight. For a given $L = (L_1, L_2, \cdots, L_n)$, we can derive the corresponding S_1, S_2, and l^*, and define:

$$Y := \begin{cases} \frac{1}{2}(\frac{\sum_{i=1}^{l^*} L_i}{L_{l^*+1}} - n + l^* + 1), & \text{if } l^* = \max\{S_1\}, \\ 1 & \text{if } l^* = \min\{S_2\}. \end{cases}$$

If $l^* = \max\{S_1\}$, $(n - l^* - 1)L_{l^*+1} < \sum_{i=1}^{l^*} L_i$, and thus Y is always positive. We then construct a utility set $U = \{u \,|\, 0 \leq u_i \leq L_i, \ \sum_{i=1}^n c_i u_i \leq 1\}$, where

$$c_i = \begin{cases} \frac{Y}{l^* L_i} & i = 1, \cdots, l^*, \\ \frac{1}{L_i} & i = l^* + 1, \cdots, n. \end{cases}$$

Now, recall that to obtain the MMF solution, we initially maximize the minimum ratio of players' utilities to their corresponding maximum achievable utilities, i.e., $\max_{u \in U} \min_{i \in N} u_i/L_i$. In the above specific example, the solution to this max-min optimization problem can be shown to be $\bar{u} := \{L_1/(Y + n - l^*), L_2/(Y + n - l^*), \cdots, L_n/(Y + n - l^*)\}$, at which, obviously, all the ratios of the players utilities to their maximum achievable utilities are equal. Since at $\bar{u}$, the constraint $\sum_{i=1}^l c_i u_i \leq 1$ in U is satisfied as equality, and since all $c_i' s$ are positive, any increase in the utility of any of the players will violate this constraint. Therefore, $\bar{u}$ must be the MMF solution.

Furthermore, since the utilitarian solution, u^*, for this example is $u^* = \{L_1, \cdots, L_{l^*}, (1 - Y)L_{l^*+1}, 0, \cdots, 0\}$, the price of MMF for this example is

$$\begin{aligned} POF(U; MMF) &= 1 - \frac{\sum_{i=1}^n u_i^{MMF}}{\sum_{i=1}^n u_i^*} \\ &= 1 - \frac{\frac{\sum_{i=1}^n L_i}{Y + n - l^*}}{\sum_{i=1}^{l^*} L_i + (1 - Y)L_{l^*+1}} = \begin{cases} 1 - \frac{4L_{l^*+1} \sum_{i=1}^n L_i}{(\sum_{i=1}^{l^*} L_i + (n - l^* + 1)L_{l^*+1})^2} & l^* \in S_1, \\ 1 - \frac{\sum_{i=1}^n L_i}{\sum_{i=1}^{l^*} L_i (n - l^* + 1)} & l^* \in S_2. \end{cases} \end{aligned}$$

■

4.4. The worst case of maximum achievable utilities

Similar to proportional fairness, let $UB(L; n, MMF)$ denote the set of all possible upper bounds with given number of players, n.

PROPOSITION 8. *Denote by $L_i \in (0,1]$ the maximum achievable utility of player i, and assume, without loss of generality, that $L_1 \geq L_2 \geq \cdots \geq L_n$. For all possible values of maximum achievable utilities, the supremum of the upper bound of the price of max-min fairness is:*

$$\sup_{L: L_i \in (0,1], L_1 \geq \cdots \geq L_n} UB(L; n, MMF) = 1 - \frac{1}{n}.$$

Proof.

(1) If $S_1 \neq \emptyset$,

$$POF(U; MMF) \leq 1 - \frac{4L_{l^*+1} \sum_{i=1}^n L_i}{(\sum_{i=1}^{l^*} L_i + (n - l^* + 1)L_{l^*+1})^2}, \qquad \text{where } l^* = \max\{S_1\};$$

(2) If $S_1 = \emptyset$,

$$POF(U; MMF) \leq 1 - \frac{\sum_{i=1}^n L_i}{\sum_{i=1}^{l^*} L_i (n - l^* + 1)}, \qquad \text{where } l^* = \min\{S_2\},$$

where $S_1 := \{l \in \mathbb{N}^+ : (n - l - 1)L_{l+1} < \sum_{i=1}^l L_i \leq (n - l + 1)L_{l+1}\}$, $S_2 := \{l \in \mathbb{N}^+ : \sum_{i=1}^l L_i > (n - l + 1)L_{l+1}\}$.

From Theorem 3, $UB(L; n, MMF)$ has the following two possible expressions:

$$UB(L; n, MMF) = 1 - \frac{4L_{l^*+1} \sum_{i=1}^n L_i}{(\sum_{i=1}^{l^*} L_i + (n - l^* + 1)L_{l^*+1})^2}, \tag{39a}$$

$$\text{If } (n - l^* - 1)L_{l^*+1} < \sum_{i=1}^{l^*} L_i \leq (n - l^* + 1)L_{l^*+1};$$

$$UB(L; n, MMF) = 1 - \frac{\sum_{i=1}^n L_i}{\sum_{i=1}^{l^*} L_i (n - l^* + 1)}, \tag{39b}$$

$$\text{If } \sum_{i=1}^{l^*} L_i > (n - l^* + 1)L_{l^*+1}.$$

(1) If $(n - l^* - 1)L_{l^*+1} < \sum_{i=1}^{l^*} L_i \leq (n - l^* + 1)L_{l^*+1}$, it follows from (39a) that

$$UB(L; n, MMF) = 1 - \frac{4L_{l^*+1} \sum_{i=1}^n L_i}{(\sum_{i=1}^{l^*} L_i + (n - l^* + 1)L_{l^*+1})^2} \leq 1 - \frac{4L_{l^*+1}(\sum_{i=1}^{l^*} L_i + L_{l^*+1})}{(\sum_{i=1}^{l^*} L_i + (n - l^* + 1)L_{l^*+1})^2}. \tag{40}$$

For convenience, let

$$q(x) = \frac{4L_{l^*+1}(x + L_{l^*+1})}{(x + (n - l^* + 1)L_{l^*+1})^2}.$$

The derivative of $q(x)$ is:

$$q'(x) = \frac{4L_{l^*+1}((n - l^* - 1)L_{l^*+1} - x)}{(x + (n - l^* + 1)L_{l^*+1})^3} < 0. \qquad \text{(By the condition of (39a))}$$

Thus,

$$q(x) \geq \frac{4L_{l^*+1}^2 (n - l^* + 2)}{4(n - l^* + 1)^2 L_{l^*+1}^2} = \frac{(n - l^* + 2)}{(n - l^* + 1)^2} \geq \frac{n+1}{n^2} > \frac{1}{n},$$

where the second-to-last inequality follows since $(n-l+2)/(n-l+1)$ is increasing with $l\in[1,n-1]$. Thus,

$$UB(L;n,MMF)\le 1-q(x)<1-\frac{1}{n}.$$

(2) If $\sum_{i=1}^{l^*}L_i>(n-l^*+1)L_{l^*+1}$, it follows from (39b) that

$$\begin{aligned}UB(L;n,MMF)&=1-\frac{\sum_{i=1}^{n}L_i}{\sum_{i=1}^{l^*}L_i(n-l^*+1)}<1-\frac{\sum_{i=1}^{l^*}L_i}{\sum_{i=1}^{l^*}L_i(n-l^*+1)}\\&=1-\frac{1}{n-l^*+1}\le 1-\frac{1}{n}.\end{aligned}\tag{41}$$

Thus, we have $UB(L;n,MMF)<1-1/n$.

We next prove that $\sup UB(L;n,PF)=1-1/n$. Indeed, if there exists a strictly smaller upper bound $1-1/n-\delta$, where $\delta>0$, let $L_1^*=L>0, L_i^*=\epsilon$ for $i=2,3,\dots,n$. Then, there exists a small $\epsilon=nL\delta/(2(n-1))$ such that the upper bound of the price of MMF with the given L^* is greater than $1-1/n-\delta$. The condition $\sum_{i=1}^{l}L_i^*>(n-l^*+1)L_{l+1}^*$ is satisfied for all possible l when ϵ is close to 0, thus, $S_1=\emptyset$, $l^*=1$, and

$$UB(L^*;n,MMF)=1-\frac{L+(n-1)\epsilon}{\sum_{i=1}^{l^*}L_i(n-l^*+1)}\ge 1-\frac{L+(n-1)\epsilon}{nL}=1-\frac{1}{n}-\frac{\delta}{2}>1-\frac{1}{n}-\delta.$$

■

Remark 2. We note that one can also easily show that for a fixed number of players, n, the supremum of the BFT bound for the price of max-min fairness over all maximum achievable utilities is $1-4/(n+1)^2$, which is strictly larger than our bound for all $n\ge 2$.

4.5. The improvement on the upper bound

The BFT bound for the price of max-min fairness was proven to be tight when all players have equal maximum achievable utilities. We claim that if $L_1=L_2=\cdots=L_n$, then our upper bound, given by Theorem 3, reduces to the BFT bound.

To show that, denote $\overline{L}:=L_1=L_2=\cdots=L_n$. We have $S_1=\{l\in\mathbb{N}^+:(n-l-1)L_{l+1}<\sum_{i=1}^{l}L_i\le(n-l+1)L_{l+1}\}=\{l\in\mathbb{N}^+:(n-1)/2<l\le(n+1)/2\}\ne\emptyset$.

If n is odd, then $(n+1)/2$ is an integer and $l^*=max\{S_1\}=(n+1)/2$. Then the upper bound (35) is

$$1-\frac{4L_{l^*+1}\sum_{i=1}^{n}L_i}{(\sum_{i=1}^{l^*}L_i+(n-l^*+1)L_{l^*+1})^2}=1-\frac{4n\overline{L}^2}{(n+1)^2\overline{L}^2}=1-\frac{4n}{(n+1)^2}.$$

If n is even, then $n/2$ is an integer and $l^*=max\{S_1\}=n/2$. Then the upper bound (35) is

$$1-\frac{4L_{l^*+1}\sum_{i=1}^{n}L_i}{(\sum_{i=1}^{l^*}L_i+(n-l^*+1)L_{l^*+1})^2}=1-\frac{4n\overline{L}^2}{(n+1)^2\overline{L}^2}=1-\frac{4n}{(n+1)^2},$$

which is equal to the BFT bound for the price of max-min fairness when all maximum achievable utilities are equal (BFT 2011).

When players have unequal maximum achievable utilities, our bound strictly improves upon the BFT bound. To explore the extent of improvement, as in the PF case, we have considered a resource allocation problem with 9 players, and we have generated randomly 100 sets of maximum achievable utility vectors L. Specifically, in the t^{th} group, each element of the vector L is randomly generated from an independent truncated normal distribution with mean of 1, and standard deviations $\sigma_t = 0.01(t-1), t = 1, 2, \dots, 100$. The normal distribution is truncated to ensure all values generated are positive. We then use the data sets to compute the BFT bound and our bound. The results are presented in Figure 5. Note that for a sample variance of zero, both bounds coincide and are equal to 0.64, which is equal to the bound derived by BFT (2011) for the equal maximum achievable utilities case. But, as is evident from Figure 5, as the sample variance increases, the BFT bound increases at a significantly faster rate than our bound, especially when the sample variance of the maximum achievable utilities is not too small.

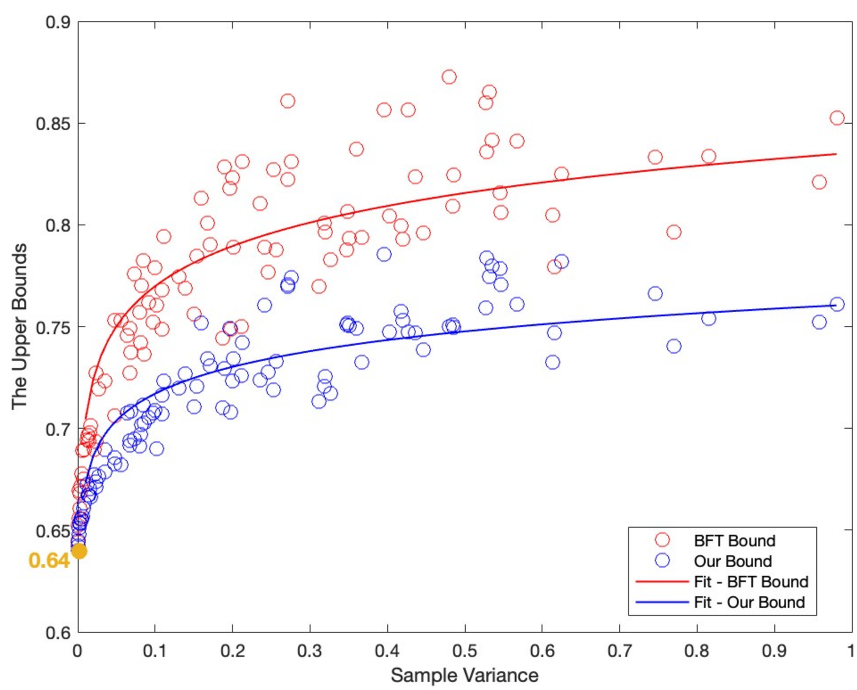


Figure 5 **The two upper bounds of the price of MMF.**

5. Conclusion

We have derived tight bounds for the price of proportional fairness for all postive integers n and for both the equal and the unequal maximum achievable utilities cases. As such, we improved on the bounds derived by BFT (2011). Specifically, their bounds for the price of proportional fairness are tight only when $\sqrt{n}$ is an integer and all players have equal maximum achievable utilities.

We have also studied the MMF criterion, which is in the spirit of Rawlsian justice and is motivated by the Kalai-Smorodinsky solution for the two-player case (see BFT 2011). For this fairness criterion, BFT (2011) have derived a tight bound for the price of MMF only when the maximum achievable utilities for all players are equal. By contrast, we have derived a tight bound for the price of MMF for the unequal maximum utilities case, which reduces to the tight bound in BFT (2011) for the case of equal maximum achievable utilities.

For both fairness notions, we have also studied the sensitivity of our bounds and BFT bounds to the variabilities of the maximum achievable utilities. Both bounds increase with the variability of the maximum achievable utilities, but the BFT bounds increase at a faster rate. In addition, in the worst case of maximum achievable utilities, we have proven that our bounds for both the PF and MMF criteria are $1-1/n$. By comparison, as we have noted, the BFT bound in the worst case for the PF criterion is also $1-1/n$, but for the MMF criterion it is $1-4/(n+1)^2$.

Endnotes

1. See https://optn.transplant.hrsa.gov/data/
2. The bound for the POF for the PF criterion is tight when the maximum achievable utilities have a special structure.

References

Alessandro Agnetis, Bo Chen, Gaia Nicosia, and Andrea Pacifici. Price of fairness in two-agent single-machine scheduling problems. *European Journal of Operational Research*, 276(1):79–87, 2019. ISSN 0377-2217. doi: https://doi.org/10.1016/j.ejor.2018.12.048. URL `https://www.sciencedirect.com/science/article/pii/S0377221719300025`.

Baris Ata, Yichuan Ding, and Stefanos Zenios. An achievable-region-based approach for kidney allocation policy design with endogenous patient choice. *Manufacturing & Service Operations Management*, 23 (1):36–54, 2021.

Cynthia Barnhart, Dimitris Bertsimas, Constantine Caramanis, and Douglas Fearing. Equitable and efficient coordination in traffic flow management. *Transportation science*, 46(2):262–280, 2012.

Dimitris Bertsimas, Vivek F Farias, and Nikolaos Trichakis. The price of fairness. *Operations research*, 59 (1):17–31, 2011.

Dimitris Bertsimas, Vivek F. Farias, and Nikolaos Trichakis. On the efficiency-fairness trade-off. *Management Science*, 58(12):2234–2250, 2012. doi: 10.1287/mnsc.1120.1549. URL `https://doi.org/10.1287/mnsc.1120.1549`.

Dimitris Bertsimas, Vivek F Farias, and Nikolaos Trichakis. Fairness, efficiency, and flexibility in organ allocation for kidney transplantation. *Operations Research*, 61(1):73–87, 2013.

Israel David and Uri Yechiali. One-attribute sequential assignment match processes in discrete time. *Operations research*, 43(5):879–884, 1995.

Cyrus Derman, Gerald J Lieberman, and Sheldon M Ross. A sequential stochastic assignment problem. *Management Science*, 18(7):349–355, 1972.

Stephanie R Earnshaw, Katherine Hicks, Anke Richter, and Amanda Honeycutt. A linear programming model for allocating hiv prevention funds with state agencies: a pilot study. *Health Care Management Science*, 10(3):239–252, 2007.

Jamie Fairbrother, Konstantinos G Zografos, and Kevin D Glazebrook. A slot-scheduling mechanism at congested airports that incorporates efficiency, fairness, and airline preferences. *Transportation Science*, 54(1):115–138, 2020.

Farhad Hasankhani and Amin Khademi. Proportionally fair organ transplantation allocation rules. *IISE Transactions*, 54(12):1131–1142, 2022.

Haruo Imai. Individual monotonicity and lexicographic maxmin solution. *Econometrica: Journal of the Econometric Society*, pages 389–401, 1983.

Yu Jiang, Zhitao Hu, Zhenyu Liu, and Honghai Zhang. Optimization of multi-objective airport gate assignment problem: considering fairness between airlines. *Transportmetrica B: Transport Dynamics*, pages 1–15, 2022.

Ehud Kalai and Meir Smorodinsky. Other solutions to Nash's bargaining problem. *Econometrica: Journal of the Econometric Society*, pages 513–518, 1975.

Özlem Karsu and Alec Morton. Inequity averse optimization in operational research. *European journal of operational research*, 245(2):343–359, 2015.

Frank P Kelly, Aman K Maulloo, and David Kim Hong Tan. Rate control for communication networks: shadow prices, proportional fairness and stability. *Journal of the Operational Research society*, 49(3): 237–252, 1998.

George Kozanidis. Solving the linear multiple choice knapsack problem with two objectives: profit and equity. *Computational Optimization and Applications*, 43(2):261–294, 2009.

Sheng Liu and Nooshin Salari. Fairness in accessibility of public service facilities. *Available at SSRN 4200501*, 2022.

Hanan Luss. On equitable resource allocation problems: A lexicographic minimax approach. *Operations Research*, 47(3):361–378, 1999. doi: 10.1287/opre.47.3.361. URL `https://doi.org/10.1287/opre.47.3.361`.

Yuanzheng Ma, Tong Wang, and Huan Zheng. On fairness and efficiency in nonprofit operations: Dynamic resource allocations. *Production and Operations Management*, 32(6):1778–1792, 2023.

R. Mazumdar, L.G. Mason, and C. Douligeris. Fairness in network optimal flow control: optimality of product forms. *IEEE Transactions on Communications*, 39(5):775–782, 1991. doi: 10.1109/26.87140.

John F Nash Jr. The bargaining problem. *Econometrica: Journal of the econometric society*, pages 155–162, 1950.

Quan Nguyen, Sanmay Das, and Roman Garnett. Scarce societal resource allocation and the price of (local) justice. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35, pages 5628–5636, 2021.

Colm O'Cinneide, Bernd Scherer, and Xiaodong Xu. Pooling trades in a quantitative investment process. *The Journal of Portfolio Management*, 32(4):33–43, 2006.

John Rawls. A theory of justice. In *Ethics*, pages 229–234. Routledge, 2004.

Rhonda Righter. A resource allocation problem in a random environment. *Operations Research*, 37(2): 329–338, 1989.

Alvin E Roth. Axiomatic models of bargaining. *Lecture Notes in Economics and Mathematical Systems*, 170, 1979a.

Alvin E Roth. An impossibility result concerning n-person bargaining games. *International Journal of Game Theory*, 8(3):129–132, 1979b.

Sarah M Ryan and Jumpol Vorasayan. Allocating work in process in a multiple-product CONWIP system with lost sales. *International Journal of Production Research*, 43(2):223–246, 2005.

Amartya Sen. *On economic inequality*. Oxford university press, 1997.

Xuanming Su and Stefanos A Zenios. Patient choice in kidney allocation: A sequential stochastic assignment model. *Operations research*, 53(3):443–455, 2005.

Xuanming Su and Stefanos A Zenios. Recipient choice can address the efficiency-equity trade-off in kidney transplantation: A mechanism design model. *Management science*, 52(11):1647–1660, 2006.

Ayten Turkcan, Bo Zeng, Kumar Muthuraman, and Mark Lawley. Sequential clinical scheduling with service criteria. *European Journal of Operational Research*, 214(3):780–795, 2011.

Angel Udías, David Ríos Insua, Javier Cano, and Hocine Fellag. Cost-efficient equitable water distribution in algeria: a bicriteria fair division problem with network constraints. *International Transactions in Operational Research*, 19(5):675–693, 2012.

Thomas Vossen, Michael Ball, Robert Hoffman, and Michael Wambsganss. A general approach to equity in traffic flow management and its application to mitigating exemption bias in ground delay programs. *Air Traffic Control Quarterly*, 11(4):277–292, 2003.

Yinyu Ye. Competitive communication spectrum economy and equilibrium. *Journal of the Operations Research Society of China*, 2(1):1–16, 2014.

H Peyton Young. *Equity: in theory and practice*. Princeton University Press, 1995.

Stefanos A Zenios, Glenn M Chertow, and Lawrence M Wein. Dynamic allocation of kidneys to candidates on the transplant waiting list. *Operations Research*, 48(4):549–569, 2000.

Yubai Zhang, Zhao Zhang, and Zhaohui Liu. The price of fairness for a two-agent scheduling game minimizing total completion time. *Journal of Combinatorial Optimization*, pages 1–19, 2020.

EC.1. Proof of Claims 1 - 6

CLAIM 1 *Define* $m := l(c^*)$. *Then,* $c_i^* = 1$ *for* $i = m+2, \cdots, n$.

Proof. Suppose, on the contrary, that there exists $k \in \{m+2, \cdots, n\}$ such that $c_k^* < 1$. Then let $c_i' = c_i^*$ for $i \leq k-1$ and $c_i' = 1$ for $i \geq k$. By construction, c' satisfies (10b). Since the first $m+1$ terms of c' and c^* coincide, c' also satisfies (10c) and (10d), and we conclude that c' is feasible for problem (10).

Since the objective function (10a) is a decreasing function of the coefficients c_k for all $k \geq m+2$, c' attains a strictly lower objective value than c^*, which contradicts the optimality of c^*. ■

CLAIM 2 $c_1^* = c_2^* = \cdots = c_m^*$.

Proof. Suppose, on the contrary, Claim 2 does not hold. Let $c_i'' = \left(\sum_{i=1}^m c_i^*\right)/m$ for all $i \leq m$, and let $c_i'' = c_i^*$ for all $i \geq m+1$. Since $\sum_{i=1}^m c_i^* = \sum_{i=1}^m c_i''$, c'' is feasible to (10).

With Claim 1, the objective function in problem (10) can be reformulated as

$$\tilde{f}_1(c_1, \ldots, c_{m+1}) := \frac{\sum_{i=1}^{m+1} \frac{1}{c_i} + n - m - 1}{m + \frac{1 - \sum_{i=1}^m c_i}{c_{m+1}}}.$$

As the harmonic mean is always not larger than the arithmetic mean, we have

$$\begin{aligned}
\tilde{f}_1(c_1^*, \ldots, c_{m+1}^*) = \frac{\sum_{i=1}^m \frac{1}{c_i^*} + \frac{1}{c_{m+1}^*} + n - m - 1}{m + \frac{1 - \sum_{i=1}^m c_i^*}{c_{m+1}^*}} &\geq \frac{\frac{m^2}{\sum_{i=1}^m c_i^*} + \frac{1}{c_{m+1}''} + n - m - 1}{m + \frac{1 - \sum_{i=1}^m c_i''}{c_{m+1}''}} \\
&= \frac{\sum_{i=1}^m \frac{1}{c_i''} + \frac{1}{c_{m+1}''} + n - m - 1}{m + \frac{1 - \sum_{i=1}^m c_i''}{c_{m+1}''}} = \tilde{f}_1(c_1'', \ldots, c_{m+1}''),
\end{aligned}$$

where equality holds only when $c^* = c''$. Therefore, if $c^* \neq c''$, c'' yields a strictly lower objective value than c^*, which contradicts the optimality of c^*. ■

CLAIM 3 $c_{m+1}^* = 1$.

Proof. With Claims 1 and 2, the objective function in (10) can be reformulated as

$$g(d, c_{m+1}) := \frac{\frac{m}{d} + \frac{1}{c_{m+1}} + n - m - 1}{m + \frac{1 - md}{c_{m+1}}}.$$

The partial derivative of $g(d, c_{m+1})$ with respect to c_{m+1} is given by

$$\frac{\partial g(d, c_{m+1})}{\partial c_{m+1}} = \frac{1}{(mc_{m+1} + 1 - md)^2} \left((n - m - 1 + \frac{m}{d})(1 - md) - m \right),$$

and note that its value is independent of c_{m+1}. That is, $\partial g(d,c_{m+1})/\partial c_{m+1}$ is either always positive, or always negative for all $c_{m+1}\in[\max\{1-md,d\},1]$. Thus, $g(d,c_{m+1})$ attains its minimum at one of the two end points of the feasible interval for c_{m+1}. To figure out which end point it attains a minimum, consider the following three cases.

Case 1. If $\partial g(d,c_{m+1})/\partial c_{m+1}\le 0$, then $g(d,c_{m+1})$ reaches its minimum when $c_{m+1}=1$.

Case 2a. If $\partial g(d,c_{m+1})/\partial c_{m+1}\ge 0$ and $d\le 1/(m+1)$, then $1-md\ge d$. Recall that $m=l(c^*)$, thus $c^*_{m+1}\in(1-md,1]$ due to (10d), and $g(d,c^*_{m+1})>g(d,1-md)$. However, in this case, for a feasible coefficient vector $\overline{c}$, whose first m components coincide with the first m components of c^* and $\overline{c}_{m+1}=1-md$, we have that $f_1(c^*)=g(d,c^*_{m+1})>g(d,1-md)=f_1(\overline{c})$, contradicting the optimality of c^*. So this case would never happen.

Case 2b. If $\partial g(d,c_{m+1})/\partial c_{m+1}\ge 0$ and $d>1/(m+1)$, then $1-md<d$. Now $c^*_{m+1}\in[d,1]$ due to (10b). However, similar to Case 2a, for a feasible coefficient vector $\overline{c}$ with $\overline{c}_1=\overline{c}_2=\cdots=\overline{c}_{m+1}=1/(m+1)$, we have $f_1(c^*)=g(d,c^*_{m+1})\ge g(d,d)=m+1+(n-m-1)d>m+1+(n-m-1)/(m+1)=f_1(\overline{c})$, again, contradicting the optimality of c^*. So Case 2b would never happen.

From the above analysis, we conclude that the only viable case is Case 1, at which $c^*_{m+1}=1$ and $\partial g(d,c_{m+1})/\partial c_{m+1}\le 0$. ■

Claim 4 *When $m\ge 2$, we have $d^*=1/m$.*

Proof. Since $c^*_{m+1}=1$, we let $h(d)=g(d,1)=(m/d+n-m)/(m+1-md)$. The derivative of $h(d)$ is given by

$$h'(d)=\frac{m(n-m)}{(m+1-md)^2d^2}\left(d^2+\frac{2m}{n-m}d-\frac{m+1}{n-m}\right).$$

Note that the sign of $h'(d)$ depends on the expression in the last bracket, denoted by $h'_0(d)$. Clearly, $h'_0(d)$ increases with d. Now, we want to show that $h'_0(1/m)\le 0$. Indeed, if it is true, then $h'_0(d)\le 0$ for all $d\le 1/m$, and thus $h(d)$ reaches its minimum at $d=1/m$.

Suppose, on the contrary, that $h'_0(1/m)>0$, which implies that

$$m^2+n-m>m^3. \tag{EC.1}$$

Let d^* denote the positive root of $h'_0(d)=0$. Then, the objective function, $h(d)$, reaches its minimum at d^*, and we know that $d^*<1/m$.

By optimality of d^* and m, the value of $f(c)$ at $c_1=\cdots=c_m=d^*,\ c_{m+1}=\cdots=c_n=1$ is smaller than that at $c_1=\cdots=c_{m+1}=1/(m+1),\ c_{m+2}=\cdots=c_n=1$, i.e.,

$$\frac{\frac{m}{d^*}+n-m}{m+1-md^*}\le\frac{(m+1)^2+n-m-1}{m+1},$$

which implies that

$$(m^2+m+n)d^{*2}-(m+2)(m+1)d^*+m+1\le 0. \tag{EC.2}$$

By (EC.1), we have

$$\begin{aligned}0 &\ge (m^2+m+n)d^{*2}-(m+2)(m+1)d^*+m+1\\ &> (m^3+2m)d^{*2}-(m+2)(m+1)d^*+m+1\\ &= (md^*-1)((m^2+2)d^*-m-1)+m(m-2).\end{aligned} \tag{EC.3}$$

Since $d^*<1/m$ and $m\ge 2$, we have $md^*-1<0, (m^2+2)d^*-m-1<2/m-1\le 0$ and $m(m-2)\ge 0$, which contradicts (EC.3).

Thus, $h_0(1/m)\le 0$ and thus $d^*=1/m$. ■

Claim 5 *When $n\ge 3$, $m\ne 1$.*

Proof. Suppose $m=1$, and assume the objective function value reaches its minimum at $\overline{d}$. Then, $\overline{d}\le 1$ by Constraint (10b). Then the objective function value at $c_1=\overline{d},\ c_2=\cdots=c_n=1$ should be smaller than or equal to that at $c_1=c_2=1/2, c_3=\cdots=c_n=1$, in which case, $m=2$ and $d=1/2$. Thus, we should have

$$\begin{aligned}&\frac{\frac{1}{\overline{d}}+n-1}{2-\overline{d}}\le\frac{2+n}{2}\\ \Rightarrow\ &\frac{\frac{1}{\overline{d}}+n-1}{2-\overline{d}}-\frac{2+n}{2}=\frac{\frac{2}{\overline{d}}+n-4}{2(2-\overline{d})}\le 0.\end{aligned} \tag{EC.4}$$

However, since $\overline{d}\le 1$ and $n\ge 3$, we have $2/\overline{d}+n-4\ge 1$ and $2(2-\overline{d})>0$, which contradicts (EC.4). Thus, $m\ne 1$ when $n\ge 3$. ■

Claim 6 *When $n=2$, $m=1, c_1^*=\sqrt{3}-1$.*

Proof. By Claim 3, $c_2^*=1$. Then the objective function is

$$f_1(c)=f_1((c_1,c_2^*))=\frac{1+c_1}{c_1(2-c_1)},$$

whose partial derivative with respect to c_1 is:

$$\frac{df_1((c_1,c_2^*))}{dc_1}=\frac{c_1^2+2c_1-2}{2(2c_1-c_1^2)^2}\begin{cases}\le 0, & c_1\in[\frac{1}{2},\sqrt{3}-1];\\ >0, & c_1\in(\sqrt{3}-1,1].\end{cases}$$

Thus, $c_1^*=\sqrt{3}-1$ and $f_1(c^*)=f_1((c_1^*,c_2^*))=(2+\sqrt{3})/4$.

■

EC.2. Proof of Lemma 1

LEMMA 1 *For any $a \in \mathbb{N}^+$ and $n \geq 2$, when $n \in [a^2, a(a+1)]$, $\Delta(n)$ increases with respect to n; when $n \in [a(a+1), (a+1)^2)$, $\Delta(n)$ decreases with respect to n. Thus, $\Delta(n)$ reaches a local maximum when $n = a(a+1)$.*

Proof.

(1) When $n \in [a^2, a(a+1)] \Leftrightarrow a \in [(\sqrt{1+4n}-1)/2, \sqrt{n}], k = \lfloor \sqrt{n} \rfloor = a, \epsilon = \sqrt{n} - a \in [0, (1+2\sqrt{n} - \sqrt{1+4n})/2]$. Thus $\Delta(n) = (\sqrt{n}-a)^2/(a(n-2\sqrt{n}+1))$, whose derivative is:

$$\Delta'(n) = \frac{(\sqrt{n}-a)(a-1)}{a\sqrt{n}(\sqrt{n}-1)^3} \geq 0.$$

(2) When $n \in [a(a+1), (a+1)^2) \Leftrightarrow a \in (\sqrt{n}-1, (\sqrt{1+4n}-1)/2], k = \lfloor \sqrt{n} \rfloor = a, \epsilon = \sqrt{n} - a \in [(1+2\sqrt{n} - \sqrt{1+4n})/2, 1)$. Thus $\Delta(n) = (1-\sqrt{n}+a)^2/((a+1)(\sqrt{n}-1)^2)$, whose derivative is:

$$\Delta'(n) = -\frac{(1-\sqrt{n}+a)a}{(a+1)\sqrt{n}(\sqrt{n}-1)^3} < 0.$$

■

EC.3. Proof of Claims 7 - 12

CLAIM 7 *$c_i^* = 1/L_i$, for $i = l+2, \cdots, n$.*

Proof.

Recall that the objective function (18a) is

$$\frac{\sum_{i=1}^n \frac{1}{nc_i}}{\sum_{i=1}^l L_i + \frac{1-\sum_{i=1}^l c_i L_i}{c_{l+1}}}.$$

Note that $c_i^* = 1/L_i, i = l+2, \ldots, n$, is feasible for problem (18), and since $c_i, i = l+2, \cdots, n$ only appear in the nominator of the objective function, (18a), which is to be minimized, at optimality, they should attain their upper bounds.

■

CLAIM 8 *$c_1^*\sqrt{L_1} = c_2^*\sqrt{L_2} = \cdots = c_l^*\sqrt{L_l}$.*

Proof.

By Cauchy–Schwarz inequality, we have

$$\sum_{i=1}^l \frac{1}{c_i} \geq \frac{(\sum_{i=1}^l \sqrt{L_i})^2}{\sum_{i=1}^l c_i L_i}, \tag{EC.5}$$

where equality holds when $c_1^2 L_1 = c_2^2 L_2 = \cdots = c_l^2 L_l$. When adjusting the values of c_i where $i = 1, 2, \ldots, l$, while keeping $\sum_{i=1}^l c_i L_i$ constant, the denominator of the objective function (18a) remains invariant, and the numerator of the objective function (18a) reaches its minimum when the equality in (EC.5) is satisfied. So, at optimality, $c_1^*\sqrt{L_1}=c_2^*\sqrt{L_2}=\cdots=c_l^*\sqrt{L_l}$.

■

CLAIM 9 $x^*=1/L_{l^*+1}$.

Proof. Recall that

$$f_2(x,y,l)=\frac{\frac{1}{y}A(l)+\frac{1}{x}+B(l)}{M(l)+\frac{1-yA(l)}{x}},\quad A(l)=\sum_{i=1}^{l}\sqrt{L_i},\quad B(l)=\sum_{i=l+2}^{n}L_i,\quad M(l)=\sum_{i=1}^{l}L_i.$$

Then, the partial derivative of $f_2(x,y,l)$ with respect to x, is:

$$\frac{\partial f_2(x,y,l)}{\partial x}=\frac{\frac{1}{y}A(l)-yA(l)B(l)-M(l)+B(l)-(A(l))^2}{(xM(l)+1-yA(l))^2}.$$

Note that since, by definition, $A(l)$ and $B(l)$ are positive, the numerator of $\partial f_2(x,y,l)/\partial x$, which is independent of x, strictly decreases with y. Moreover, $\partial f_2(x,y,l)/\partial x>0$ when $y=0+$ and $\partial f_2(x,y,l)/\partial x<0$ when $y=1/A(l)$. As a result, there exists $\overline{y}(l)\in(0,1/A(l))$ at which $\partial f_2(x,y,l)/\partial x=0$. Thus, $\partial f_2(x,y,l)/\partial x>0$ when $y\in(0,\overline{y}(l))$, and $\partial f_2(x,y,l)/\partial x\le 0$ when $y\in[\overline{y}(l),1/A(l)]$. Next we prove that at optimality, $y^*\in[\overline{y}(l^*),1/A(l^*)]$. Therefore, the objective function is minimized when x reaches its upper limit, $1/L_{l^*+1}$, which proves the claim.

We prove that $y^*\in[\overline{y}(l^*),1/A(l^*)]$ by contradiction. Suppose, on the contrary, that $y^*\in(0,\overline{y}(l^*))$, and let us focus on $y\in(0,\overline{y}(l^*))$, for which we have that $\partial f_2(x,y,l^*)/\partial x>0$. Since, by (20d), $x>(1-yA(l^*))/L_{l^*+1}$, we have

$$f_2(x,y,l^*)>f_2(\frac{1-yA(l^*)}{L_{l^*+1}},y,l^*)=\frac{B(l^*)+\frac{A(l^*)}{y}+\frac{L_{l^*+1}}{1-yA(l^*)}}{M(l^*+1)}.$$

The partial derivative of $f_2((1-yA(l^*))/L_{l^*+1},y,l^*)$ with respect to y has the following form,

$$\frac{\partial f_2(\frac{1-yA(l^*)}{L_{l^*+1}},y,l^*)}{\partial y}=\frac{A(l^*)\cdot((L_{l^*+1}-A(l^*))y^2+2A(l^*)\cdot y-1)}{M(l^*+1)\cdot y^2\cdot(A(l^*)\cdot y-1)^2},$$

and it is negative when $y\in(0,1/A(l^*+1))$, and it is positive when $y\in(1/A(l^*+1),1/A(l^*)]$, implying that $f_2((1-yA(l^*))/L_{l^*+1},y,l^*)$ is minimized at $y=1/A(l^*+1)$. Thus, for all feasible y, we have

$$\begin{aligned}f_2(\tfrac{1-yA(l^*)}{L_{l^*+1}},y,l^*)&\ge f_2(\tfrac{1-\frac{A(l^*)}{A(l^*+1)}}{L_{l^*+1}},\tfrac{1}{A(l^*+1)},l^*)=\tfrac{B(l^*)+(A(l^*+1))^2}{M(l^*+1)}\\&=\tfrac{(A(l^*+1))^2+L_{l^*+2}+B(l^*+1)}{M(l^*+1)}=f_2(\tfrac{1}{L_{l^*+2}},\tfrac{1}{A(l^*+1)},l^*+1).\end{aligned}$$

Thus, as long as $y^*\in(0,\overline{y}(l^*))$, for all feasible x, we have $f_2(x,y,l^*)>f_2((1-yA(l^*))/L_{l^*+1},y,l^*)\ge f_2(1/L_{l^*+2},1/A(l^*+1),l^*+1)$. Therefore, any $l^*\le n-2$ is sub-optimal as a smaller objective can be attained by replacing l^* with l^*+1, contradicting the optimality of l^*. We conclude that the only possible case for $y^* \in (0, \overline{y}(l^*))$ is $l^* = n-1$, since, in this case, replacing l^* with l^*+1, would lead to $l^* = n$, which is infeasible.

Next, we prove that when $l^* = n-1$, $y^* \in (0, \overline{y}(l^*))$ is also impossible, and $x^* = 1/L_n, y^* = 1/A(n-1)$.

Since $B(l) = \sum_{l+2}^{n} L_i$, we have $B(n-1) = 0$, and $\partial f_2(x,y,n-1)/\partial x = (A(n-1)/y - M(n-1) - (A(n-1))^2)/(xM(n-1)+1-yA(n-1))^2$. Thus, $\overline{y}(n-1)$, as the solution to $\partial f_2(x,y,n-1)/\partial x = 0$, has the following expression,

$$\overline{y}(n-1) = \frac{A(n-1)}{(A(n-1))^2 + M(n-1)}.$$

From the previous analysis, $f_2((1-yA(l^*))/L_{l^*+1}, y, l^*)$ decreases in y when $y \in [0, 1/A(l^*+1)] = [0, 1/A(n)]$. We observe that

$$\overline{y}(n-1) = \frac{A(n-1)}{(A(n-1))^2 + M(n-1)} \le \frac{A(n-1)}{(A(n-1))^2 + A(n-1)\sqrt{L_n}} = \frac{1}{A(n-1)+\sqrt{L_n}} = \frac{1}{A(n)}.$$

Thus, $f_2((1-yA(n-1))/L_n, y, n-1)$ decreases over $[0, \overline{y}(n-1)]$. Consequently,

$$\begin{aligned} f_2(x,y,n-1) &> f_2((1-yA(n-1))/L_n, y, n-1) \ge f_2((1-\overline{y}(n-1)\cdot A(n-1))/L_n, \overline{y}(n-1), n-1) \\ &= \frac{(A(n-1))^2 + M(n-1)}{M(n-1)} \ge \frac{(A(n-1))^2 + L_n}{M(n-1)} = f_2(1/L_n, 1/A(n-1), n-1), \end{aligned}$$

and we conclude that, given that $l^* = n-1, x^* = 1/L_n$, $y^* = 1/A(n-1)$ is optimal. However, we reached a contradiction, since $y^* \le \overline{y}(n-1) \le 1/(A(n-1)+\sqrt{L_n}) < 1/A(n-1)$, which implies that $y^* \ge \overline{y}(l^*)$ and thus $x^* = 1/L_{l^*+1}$. ■

Claim 10 *(1) When $l^* = 1$ with $(B(0)+L_1)L_2 \le L_1^2$, or when $l^* \ge 2$, $y^* = 1/A(l^*)$.*
(2) When $l^* = 1$ with $(B(0)+L_1)L_2 > L_1^2$, $y^* = \tilde{y} := (-\sqrt{L_1} + \sqrt{B(0)+L_1+B(0)L_1/L_2})/B(0)$.

Proof.

According to Claim 9, $x^* = 1/L_{l^*+1}$, and thus, the objective function of (20) can be written as:

$$g(y,l) = \frac{\frac{A(l)}{y} + B(l-1)}{M(l+1) - yA(l)L_{l+1}}.$$

The partial derivative of $g(y,l)$ with respect to y has the following form:

$$\frac{\partial g(y,l)}{\partial y} = \frac{A(l)}{y^2(M(l+1) - A(l)L_{l+1}y)^2}\left((B(l)L_{l+1} + L_{l+1}^2)y^2 + 2A(l)L_{l+1}y - M(l+1)\right).$$

Let $\tilde{y}$ denote the positive root of $\partial g(y,l)/\partial y = 0$. Then by the quadratic formula,

$$\tilde{y} = \frac{-A(l)L_{l+1} + \sqrt{(A(l))^2 L_{l+1}^2 + L_{l+1}M(l+1)B(l-1)}}{B(l-1)L_{l+1}},$$

and the above partial derivative $\partial g(y,l)/\partial y$ is negative when $y<\tilde{y}$, and it is positive when $y>\tilde{y}$. Thus, if $\tilde{y}\ge 1/A(l)$, then, since by (20c), for all feasible y, $y\le 1/A(l)$, we have that $\partial g(y,l)/\partial y\le 0$, and thus $g(y,l)$ attains the minimum at $y=1/A(l)$. If, on the other hand, $\tilde{y}<1/A(l)$, then $\partial g(y,l)/\partial y\le 0$ for $y\in(0,\tilde{y}]$, and $\partial g(y,l)/\partial y>0$ for $y\in(\tilde{y},1/A(l)]$, implying that the objective value attains the minimum at $y=\tilde{y}$. Thus, we have $y^*=\min\{\tilde{y},1/A(l^*)\}$.

Recall that $A(l)=\sum_{i=1}^{l}\sqrt{L_i}, M(l)=\sum_{i=1}^{l}L_i, B(l)=\sum_{i=l+2}^{n}L_i$.

If $l^*=1$ with $\tilde{y}<1/A(1)\Leftrightarrow(B(0)+L_1)L_2>L_1^2$, which proves Case (2) of the claim.

If $l^*=1$ with $\tilde{y}\ge 1/A(1)\Leftrightarrow(B(0)+L_1)L_2\le L_1^2$, which proves Case (1) of the claim for $l^*=1$.

When $l^*\ge 2$, $\tilde{y}\ge 1/A(l^*)\Leftrightarrow(A(l^*))^2\cdot M(l^*)\ge[(A(l^*))^2+B(l^*-1)]\cdot L_{l^*+1}$. We next prove this inequality by contradiction, which would imply that $y^*=1/A(l^*)$ when $l^*\ge 2$.

So, suppose, on the contrary, that

$$(A(l^*))^2\cdot M(l^*)<[(A(l^*))^2+B(l^*-1)]\cdot L_{l^*+1}, \tag{EC.6}$$

then $\tilde{y}<1/A(l^*)$, and we have $y^*=\tilde{y}$.

Since, by definition, the objective function, $g(y,l)$, attains its minimum at $x=1/L_{l^*+1},y=y^*,l=l^*$, its value at this point is smaller than or equal to its value at $x=1/L_{l^*+2},y=1/A(l^*+1),l=l^*+1$, i.e.,

$$\begin{aligned}
&\frac{\frac{A(l^*)}{y^*}+B(l^*-1)}{M(l^*+1)-y^*A(l^*)L_{l^*+1}}\le\frac{(A(l^*+1))^2+B(l^*)}{M(l^*+1)}\\
\Rightarrow\ &[(A(l^*))^2+2A(l^*)\sqrt{L_{l^*+1}}+B(l^*-1)]L_{l^*+1}y^{*2}-(A(l^*)+2\sqrt{L_{l^*+1}})\cdot M(l^*+1)\cdot y^*\\
&\qquad+M(l^*+1)\le 0.
\end{aligned}$$

By (EC.6), we have

$$\begin{aligned}
0\ge&[(A(l^*))^2+2A(l^*)\sqrt{L_{l^*+1}}+B(l^*-1)]L_{l^*+1}y^{*2}-(A(l^*)+2\sqrt{L_{l^*+1}})\cdot M(l^*+1)\cdot y^*\\
&+M(l^*+1)\\
>&[(A(l^*))^2M(l^*)+2A(l^*)L_{l^*+1}\sqrt{L_{l^*+1}}]y^{*2}-(A(l^*)+2\sqrt{L_{l^*+1}})M(l^*+1)\cdot y^*+M(l^*+1)\\
=&(A(l^*)y^*-1)[(A(l^*)M(l^*)+2L_{l^*+1}\sqrt{L_{l^*+1}})y^*-M(l^*+1)]+(A(l^*)-2\sqrt{L_{l^*+1}})M(l^*).
\end{aligned} \tag{EC.7}$$

We will next show that the bottom expression in (EC.7), denoted by Q, is positive, which reveals that (EC.7) does not hold. Consider the first term in Q, which consists of the product of two terms, say P_1 and P_2. By supposition, we know $\tilde{y}=y^*<1/A(l^*)$. Thus, $P_1=A(l^*)y^*-1<0$. Next, consider the term P_2. We have $P_2=(A(l^*)M(l^*)+2L_{l^*+1}\sqrt{L_{l^*+1}})y^*-M(l^*+1)<(A(l^*)y^*-1)M(l^*)+2L_{l^*+1}\sqrt{L_{l^*+1}}y^*-L_{l^*+1}<2L_{l^*+1}\sqrt{L_{l^*+1}}/A(l^*)-L_{l^*+1}=L_{l^*+1}(2\sqrt{L_{l^*+1}}-A(l^*))/A(l^*)$.

When $l^*\ge 2$, $2\sqrt{L_{l^*+1}}-A(l^*)\le 0$. Thus, the first product in the right-hand side of Equation (EC.7) is positive. Also the second product is non-negative, which means (EC.7) does not hold. We conclude that, (EC.6) does not hold, i.e., $(A(l^*))^2\cdot M(l^*)\ge[(A(l^*))^2+B(l^*-1)]\cdot L_{l^*+1}$.

■

Recall that

$$h(l) := f_2\left(\frac{1}{L_{l+1}}, \frac{1}{A(l)}, l\right) = \frac{(A(l))^2 + L_{l+1} + B(l)}{M(l)} = \frac{(A(l))^2 + B(l-1)}{M(l)}.$$

Claim 11 *$h(l)$ is decreasing when $l \in [1, l^*]$, increasing when $l \in [l^*, n-1]$, and $l^* = \arg\min_l h(l)$ satisfies the following two inequalities if $l^* \in \{2, 3, \cdots, n-2\}$:*

$$\begin{cases} \sqrt{L_{l^*+1}}[(A(l^*))^2 + B(l^*-1)] \le 2A(l^*)M(l^*), & \text{(EC.8)} \\ \sqrt{L_{l^*}}[(A(l^*))^2 + B(l^*-1)] \ge 2M(l^*)A(l^*-1), & \text{(EC.9)} \end{cases}$$

where if $l^* = 1$, only (EC.8) *is satisfied, and if $l^* = n-1$, only* (EC.9) *is satisfied.*

Proof. Since $l \in \{1, 2, \cdots, n-1\}$, there exists a l^* minimizing $h(l)$. Thus, for the l^*, we have

$$\begin{cases} h(l^*) \le h(l^*+1), & \text{(EC.10)} \\ h(l^*) \le h(l^*-1). & \text{(EC.11)} \end{cases}$$

From (EC.10), we have

$$\begin{aligned} &\frac{(A(l^*))^2 + B(l^*-1)}{M(l^*)} \le \frac{(A(l^*+1))^2 + B(l^*)}{M(l^*+1)}, \\ \Rightarrow\ & (M(l^*) + L_{l^*+1})[(A(l^*))^2 + B(l^*-1)] \le M(l^*)[(A(l^*) + \sqrt{L_{l^*+1}})^2 + B(l^*-1) - L_{l^*+1}], \\ \Rightarrow\ & \sqrt{L_{l^*+1}}[(A(l^*))^2 + B(l^*-1)] \le 2A(l^*)M(l^*), \end{aligned}$$

which coincides with (EC.8).

Then we have

$$\begin{aligned} h(l^*+2) - h(l^*+1) = &\frac{\sqrt{L_{l^*+2}}}{M(l^*+1)\cdot M(l^*+2)} \\ &\{2A(l^*)M(l^*) + 2A(l^*)L_{l^*+1} + 2M(l^*+1)\sqrt{L_{l^*+1}} \\ &- [(A(l^*))^2 + 2A(l^*)\sqrt{L_{l^*+1}} + B(l^*-1)]\sqrt{L_{l^*+2}}\} \\ \ge &\frac{\sqrt{L_{l^*+2}}}{M(l^*+1)\cdot M(l^*+2)}\{[(A(l^*))^2 + B(l^*-1) + 2A(l^*)\sqrt{L_{l^*+1}}]\cdot \\ &(\sqrt{L_{l^*+1}} - \sqrt{L_{l^*+2}}) + 2M(l^*+1)\sqrt{L_{l^*+1}}\} \\ \ge &\, 0. \end{aligned}$$

Similarly, from (EC.11), we have

$$\begin{aligned} &\frac{(A(l^*))^2 + B(l^*-1)}{M(l^*)} \le \frac{(A(l^*-1))^2 + B(l^*-2)}{M(l^*-1)}, \\ \Rightarrow\ & (M(l^*) - L_{l^*})[(A(l^*))^2 + B(l^*-1)] \le M(l^*)[(A(l^*) - \sqrt{L_{l^*}})^2 + B(l^*-1) + L_{l^*}], \\ \Rightarrow\ & \sqrt{L_{l^*}}[(A(l^*))^2 + B(l^*-1)] \ge 2M(l^*)A(l^*-1), \end{aligned}$$

which coincides with (EC.9).

Then we have

$$
\begin{aligned}
h(l^*-2)-h(l^*-1) &= \frac{\sqrt{L_{l^*-1}}}{M(l^*-1)\cdot M(l^*-2)} \\
&\quad (2\sqrt{L_{l^*-1}}-2A(l^*-1))M(l^*-1)+ \\
&\quad \sqrt{L_{l^*-1}}[(A(l^*))^2-2A(l^*)\sqrt{L_{l^*}}+2L_{l^*}+B(l^*-1)] \\
&\geq \frac{\sqrt{L_{l^*-1}}}{M(l^*-1)\cdot M(l^*-2)}[2\sqrt{L_{l^*-1}}\cdot M(l^*-1) \\
&\quad +2A(l^*-1)\cdot(\sqrt{L_{l^*-1}}-\sqrt{L_{l^*}})\cdot\frac{M(l^*-1)}{\sqrt{L_{l^*}}}] \\
&\geq 0.
\end{aligned}
$$

■

Claim 12 *(x^*, y^*, l^*), as characterized in Claims 7-9, is feasible and thus optimal to Problem* (19)*.*

Proof. It suffices to show that the optimal solution (x^*, y^*, l^*) to (20) satisfies the two constraints in (19) that were removed from (20), that is,

$$
\frac{y^*}{\sqrt{L_{l^*}}} \leq x^*, \tag{EC.12a}
$$

$$
\frac{1}{n\sqrt{L_{l^*}}} \leq y^*. \tag{EC.12b}
$$

When $l^*=1$ with $(B(0)+L_1)L_2 \leq L_1^2$, or when $l^* \geq 2$, by Claims 9 and 10, the optimal solution for (20) is $x^* = 1/L_{l^*+1}, y^* = 1/A(l^*)$. Since $A(l^*)\sqrt{L_{l^*}} \geq L_{l^*} \geq L_{l^*+1}$, (EC.12a) holds.

Let us consider (EC.12b). Then, we have $A^2(l^*) = M(l^*) + 2\sum_{1\leq i<j\leq l^*}\sqrt{L_iL_j} \leq l^*M(l^*)$, and by (EC.9),

$$
\begin{aligned}
\frac{A(l^*-1)}{\sqrt{L_{l^*}}} \leq \frac{A^2(l^*)+B(l^*-1)}{2M(l^*)} \leq \frac{l^*M(l^*)+\frac{n-l^*}{l^*}M(l^*)}{2M(l^*)} \leq \frac{1}{2}(l^*+\frac{n}{l^*}-1) \leq n-1, \\
\Rightarrow A(l^*) \leq n\sqrt{L_{l^*}}, \qquad \Rightarrow \frac{1}{n\sqrt{L_{l^*}}} \leq y^*.
\end{aligned}
$$

Thus, (EC.12b) holds.

When $l^*=1$ with $(B(0)+L_1)L_2 > L_1^2$, by Claims 9 and 10, the optimal solution for (20) is $x^* = 1/L_2, y^* = (-\sqrt{L_1}+\sqrt{B(0)+L_1+B(0)L_1/L_2})/B(0)$. Since $y^* \leq 1/\sqrt{L_1} \leq \sqrt{L_1}/L_2$, (EC.12a) holds. For (EC.12b), since $B(0) \leq (n-1)L_2$, we have

$$
\begin{aligned}
y^* &= \frac{-\sqrt{L_1}+\sqrt{B(0)+L_1+B(0)\frac{L_1}{L_2}}}{B(0)} = \frac{1+\frac{L_1}{L_2}}{\sqrt{B(0)+L_1+B(0)\frac{L_1}{L_2}}+\sqrt{L_1}} \\
&\geq \frac{1+\frac{L_1}{L_2}}{\sqrt{(n-1)L_2+nL_1}+\sqrt{L_1}} = \frac{1}{\sqrt{L_1}}\cdot\frac{1+\frac{L_1}{L_2}}{\sqrt{(n-1)\frac{L_2}{L_1}+n}+1} \geq \frac{1}{n\sqrt{L_1}},
\end{aligned}
$$

so (EC.12b) holds. ■

EC.4. Proof of Lemma 2

LEMMA 2 *For* $n \geq 3$,

$$\min_{l\in\mathbb{N}} \frac{l^2+n-l}{l} \leq \frac{\sqrt{2n-1}+n}{2}. \tag{EC.13}$$

Proof. First consider the case where $n \geq 5$, and let $f(l) = (l^2+n-l)/l$. Then, $f(l)$ decreases monotonically when $l \in [1, \sqrt{n}]$ and increases monotonically when $l \in [\sqrt{n}, n-1]$. Recall that l is restricted to be integer and note that there must be an integer between $\sqrt{n}$ and $\sqrt{n}+1$. So we have

$$\min_{l\in\mathbb{N}} \frac{l^2+n-l}{l} \leq f(\sqrt{n}+1) = \sqrt{n} + \frac{n}{\sqrt{n}+1}.$$

Let

$$g(n) = \frac{\sqrt{2n-1}+n}{2} - \sqrt{n} - \frac{n}{\sqrt{n}+1}.$$

We first show that for $n \geq 5$, $g(n) \geq 0$, which implies that (EC.13) holds for this case. Now, the derivative of $g(n)$ is:

$$g'(n) = \frac{(n-3)\sqrt{n}-1}{\sqrt{n}(1+\sqrt{n})^2} + \frac{1}{\sqrt{2n-1}},$$

which is positive for $n \geq 5$. Thus, $g(n) \geq g(5) = (21-9\sqrt{5})/4 > 0$ for $n \geq 5$.

It remains to consider the cases when $n = 3, 4$.

When $n = 3, \min_{l\in\mathbb{N}}(l^2+n-l)/l = 2.5, (\sqrt{2n-1}+n)/2 = (3+\sqrt{5})/2 \approx 2.62$, thus (EC.13) holds.

When $n = 4, \min_{l\in\mathbb{N}}(l^2+n-l)/l = 3, (\sqrt{2n-1}+n)/2 = (4+\sqrt{7})/2 \approx 3.32$, thus (EC.13) holds.

■

EC.5. Proof of Claims 13-17

CLAIM 13 $c_i^* = 1/L_i$ *for* $i = l+2, \dots, n$.

Proof. Note first that $c_i, i = l+2, \dots, n$, solely appear in the first term of the denominator of the objective function (34a). Therefore, to minimize (34a), they should be assigned their maximum values within the feasible domain.

Next, we show that we can assume, without loss of generality, that $L_{l+2} \geq L_{l+3} \geq \cdots \geq L_n$. Indeed, suppose, on the contrary, that this assumption is not satisfied, and that $L_j < L_{j+1}$, for some $j \in [l+2, n-1]$. Then Constraints (34b) and (34c) imply that $c_j \leq c_{j+1} \leq 1/L_{j+1} < 1/L_j$. Thus, $\sum_{i=l+2}^n c_i L_i < n-l-1$. On the other hand, let $L_i' = L_i, i = 1, \dots, l+1$, and let $L_i', i = l+2, \dots, n$, denote the partial vector derived from the partial vector $(L_i), i = l+2, \dots, n$, after rearranging the components of the latter in a descending order. That is, $L_{l+2}' \geq L_{l+3}' \geq \cdots \geq L_n'$. Similarly, let $c_i' = c_i, i = 1, \dots, l+1$, and let $c_i' = 1/L_i', i = l+2, \dots, n$, which denotes a feasible solution to the optimization Problem (34) corresponding to the parameters $L_i', i = 1, \dots, n$. Thus, $\sum_{i=l+2}^n c_i' L_i' = n-l-1 > \sum_{i=l+2}^n c_i L_i$, while the other terms in the objective function remain invariant for these two sets of parameters and variables. Consequently, we have $f_3(c', l; L') \le f_3(c, l; L)$. Thus, we can assume $L_{l+2} \ge L_{l+3} \ge \cdots \ge L_n$.

This assumption ensures that $c_i = 1/L_i, i = l+2, \cdots, n$ satisfies Constraint (34b) and hence is feasible. Incorporating the above analysis that $c_i, i = l+2, \cdots, n$ should take the maximum value within the feasible domain, we have $c_i^* = 1/L_i, i = l+2, \cdots, n$.

■

CLAIM 14 *If* $A \le (n-l-1)L_{l+1}$,

$$x^* = \textit{any value in } [\frac{1}{A+L_{l+1}}, \frac{1}{L_{l+1}}], \quad y^* = 1 - x^* L_{l+1}, \quad g(x^*, y^*; l) = \frac{1}{(n-l)(A+L_{l+1})};$$

If $(n-l-1)L_{l+1} < A \le (n-l+1)L_{l+1}$,

$$x^* = \frac{1}{L_{l+1}}, \quad y^* = \frac{1}{2}\left(\frac{A}{L_{l+1}} - n + l + 1\right), \quad g(x^*, y^*; l) = \frac{4L_{l+1}}{(A + (n-l+1)L_{l+1})^2};$$

If $A > (n-l+1)L_{l+1}$,

$$x^* = \frac{1}{L_{l+1}}, \quad y^* = 1, \quad g(x^*, y^*; l) = \frac{1}{A(n-l+1)}.$$

Proof. If $A \le (n-l-1)L_{l+1}$, let x^* be any value in $[1/(A+L_{l+1}), 1/L_{l+1}]$, $y^* = 1 - x^* L_{l+1}$, then

$$g(x^*, y^*; l) = \frac{x^*}{(y^* + n - l - 1 + x^* L_{l+1})(Ax^* + 1 - y^*)} = \frac{1}{(n-l)(A+L_{l+1})}.$$

For any feasible x, y, we have

$$\begin{aligned}
g(x,y;l) - &g(x^*, y^*; l) \\
&= \frac{(1-y-xL_{l+1})x \cdot A + (n-l-1+y)(y+xL_{l+1}-1)}{(n-l)(A+L_{l+1})(y+n-l-1+xL_{l+1})(Ax+1-y)} \\
&\ge \frac{(1-y-xL_{l+1})(n-l-1)xL_{l+1} + (n-l-1+y)(y+xL_{l+1}-1)}{(n-l)(A+L_{l+1})(y+n-l-1+xL_{l+1})(Ax+1-y)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(since, by (34e), } 1-y-xL_{l+1} < 0) \\
&= \frac{(y+xL_{l+1}-1)((n-l-1)(1-xL_{l+1})+y)}{(n-l)(A+L_{l+1})(y+n-l-1+xL_{l+1})(Ax+1-y)} \\
&\ge 0.
\end{aligned}$$

If $(n-l-1)L_{l+1} < A \le (n-l+1)L_{l+1}$, let $x^* = 1/L_{l+1}$, $y^* = (A/L_{l+1} - n + l + 1)/2$, then

$$g(x^*, y^*; l) = \frac{1/L_{l+1}}{(A/L_{l+1} + n - l + 1)^2/4} = \frac{4L_{l+1}}{(A + (n-l+1)L_{l+1})^2}.$$

For any feasible x, y, we have

$$\begin{aligned}
g(x,y;l) - &g(x^*, y^*; l) \\
&= \frac{x(A - L(-3-l+n+2xL_{l+1}+2y))^2 + 4L_{l+1}(1-xL_{l+1})(xL_{l+1}+y-1)}{(A+(n-l+1)L_{l+1})^2(y+n-l-1+xL_{l+1})(Ax+1-y)} \\
&\ge \frac{4L_{l+1}(1-xL_{l+1})(xL_{l+1}+y-1)}{(A+(n-l+1)L_{l+1})^2(y+n-l-1+xL_{l+1})(Ax+1-y)} \\
&\ge 0.
\end{aligned}$$

If $A > (n-l+1)L_{l+1}$, let $x^* = 1/L_{l+1}$, $y^* = 1$, then

$$g(x^*, y^*; l) = \frac{x^*}{(y^* + n - l - 1 + x^* L_{l+1})(Ax^* + 1 - y^*)} = \frac{1}{A(n-l+1)}.$$

For any feasible x, y, we have

$$\begin{aligned} g(x,y;l) - & g(x^*,y^*;l) \\ &= \frac{(2 - y - L_{l+1}x)Ax - (y + xL_{l+1} + n - l - 1)(1-y)}{A(n-l+1)(y+n-l-1+xL_{l+1})(Ax+1-y)} \\ &> \frac{(2 - y - L_{l+1}x)(n-l+1)xL_{l+1} - (y + xL_{l+1} + n - l - 1)(1-y)}{A(n-l+1)(y+n-l-1+xL_{l+1})(Ax+1-y)} \\ &= \frac{(1-y)^2 + (1 - xL_{l+1})(xL_{l+1} + (n-l)(xL_{l+1} + y - 1))}{A(n-l+1)(y+n-l-1+xL_{l+1})(Ax+1-y)} \\ &\geq 0. \end{aligned}$$

The last inequality holds due to $1 - xL_{l+1} \geq 0$ by (37c), $xL_{l+1} + y - 1 \geq 0$ by (37e), $n \geq l+1$ by definition and $1 - y \geq 0$ by (37d).

■

Claim 15 *$p(A, L_{l+1}; l)$ is decreasing in both A and L_{l+1}.*

Proof. We first show that $p(A, L_{l+1}; l)$ decreases in A. Recall that $p(A, L_{l+1}; l)$ is defined in (38), and $A = \sum_{i=1}^{l} L_i$. Now, note that A only appears in the denominator of $p(A, L_{l+1}; l)$, and in all three ranges for which p is defined, an increase in A increases the denominator of $p(A, L_{l+1}; l)$. Thus, in each of these three ranges, p is decreasing in A.

Further, note that if $A = (n-l-1)L_{l+1}$, then $1/((n-l)(A + L_{l+1})) = 4L_{l+1}/(A + (n-l+1)L_{l+1})^2$; and if $A = (n-l+1)L_{l+1}$, then $4L_{l+1}/(A + (n-l+1)L_{l+1})^2 = 1/(A(n-l+1))$. Thus, $p(A, L_{l+1}; l)$ is continuous in A over the entire region, and we conclude that $p(A, L_{l+1}; l)$ is decreasing in A.

To prove that $p(A, L_{l+1}; l)$ decreases in L_{l+1}, we rewrite $p(A, L_{l+1}; l)$ as follows:

$$p(A, L_{l+1}; l) := \begin{cases} \frac{1}{A(n-l+1)}, & L_{l+1} < \frac{A}{n-l+1}; \\ \frac{4L_{l+1}}{(A+(n-l+1)L_{l+1})^2}, & \frac{A}{n-l+1} \leq L_{l+1} < \frac{A}{n-l-1}; \\ \frac{1}{(n-l)(A+L_{l+1})}, & L_{l+1} \geq \frac{A}{n-l-1}. \end{cases}$$

Clearly, if $L_{l+1} < A/(n-l+1)$ or $L_{l+1} \geq A/(n-l-1)$, then $p(A, L_{l+1}; l)$ is decreasing in L_{l+1}. For the region where $A/(n-l+1) \leq L_{l+1} < A/(n-l-1)$, we calculate the partial derivative of $p(A, L_{l+1}; l)$ with respect to L_{l+1},

$$\frac{\partial p}{\partial L_{l+1}} = \frac{4(A - (n-l+1)L_{l+1})}{(A + (n-l+1)L_{l+1})^3},$$

which is obviously negative in this region. Thus, $p(A, L_{l+1}; l)$ is decreasing in L_{l+1} in this region as well. Finally, it is easy to verify that $p(A, L_{l+1}; l)$ is continuous in L_{l+1}, and thus $p(A, L_{l+1}; l)$ is decreasing in L_{l+1}. ■

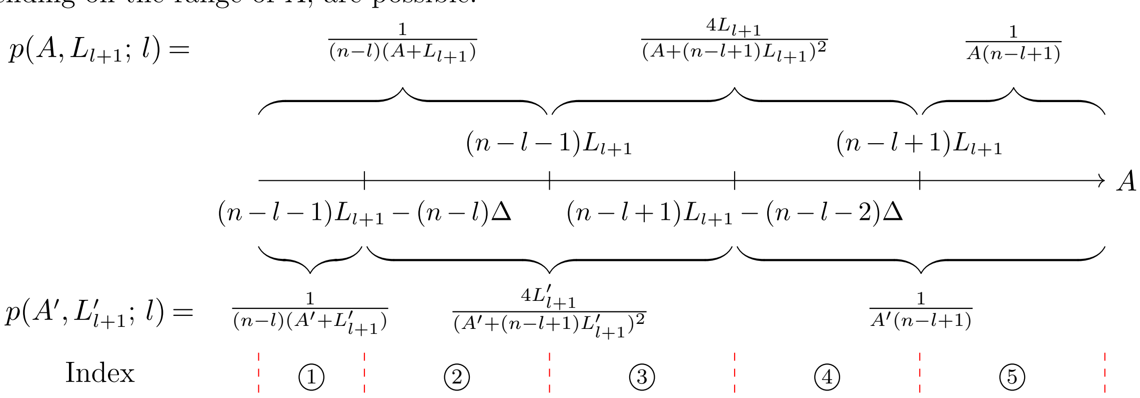
Claim 16 *The value of $p(A, L_{l+1}; l)$ is minimized with respect to the $L_i, i = 1, \ldots, n$, if they are arranged in a descending order.*

Proof. We have previously shown that the permutation of the $L_i's$ that would minimize $p(A, L_{l+1}; l)$ should satisfy: $L_1 \geq L_2 \geq \cdots \geq L_l \geq L_{l+2}$, $L_{l+1} \geq L_{l+2}$ and $L_{l+2} \geq L_{l+3} \geq \cdots \geq L_n$. Thus, what remains to show is $L_{l+1} \leq L_l$. Suppose it is not true, then we can construct L' which is in descending order and for which p attains a smaller value.

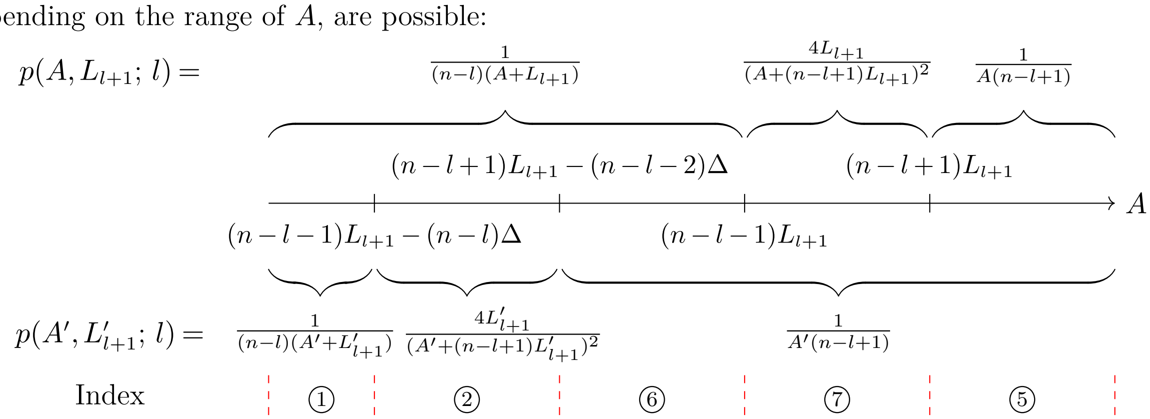
Let $\Delta = L_{l+1} - L_l > 0, L'_{l+1} = L_{l+1} - \Delta, A' = A + L_{l+1} - L'_{l+1} = A + \Delta$. The relationship among $A', (n-l-1)L'_{l+1}$ and $(n-l+1)L'_{l+1}$, as well as the relationship among $A, (n-l-1)L_{l+1}$ and $(n-l+1)L_{l+1}$, affect which segments of the piecewise functions $p(A', L'_{l+1}; l)$ and $p(A, L_{l+1}; l)$ are valid.

The following two diagrams present the valid expressions for $p(A', L'_{l+1}; l)$ and $p(A, L_{l+1}; l)$ in seven possible different cases.

(1) If $(n-l-1)L_{l+1} \leq (n-l+1)L_{l+1} - (n-l-2)\Delta$, then the following five different cases, depending on the range of A, are possible:



(2) If $(n-l-1)L_{l+1} > (n-l+1)L_{l+1} - (n-l-2)\Delta$, then the following five different cases, depending on the range of A, are possible:



Next, we proceed to analyze the seven cases identified above:

①$A \le (n-l-1)L_{l+1} - (n-l)\Delta$.

$$\begin{aligned} p(A', L'_{l+1}; l) &= \frac{1}{(n-l)(A'+L'_{l+1})} = \frac{1}{(n-l)(A+\Delta+L_{l+1}-\Delta)} \\ &= \frac{1}{(n-l)(A+L_{l+1})} = p(A, L_{l+1}; l). \end{aligned}$$

②$(n-l-1)L_{l+1} - (n-l)\Delta < A \le \min\{(n-l-1)L_{l+1}, (n-l+1)L_{l+1} - (n-l-2)\Delta\}$.

$$\begin{aligned} p(A', L'_{l+1}; l) &= \frac{4L'_{l+1}}{(A'+(n-l+1)L'_{l+1})^2} = \frac{4(L_{l+1}-\Delta)}{(A+L_{l+1}+(n-l)(L_{l+1}-\Delta))^2} \\ &\le \frac{4(L_{l+1}-\Delta)}{4(A+L_{l+1})(n-l)(L_{l+1}-\Delta)} = \frac{1}{(A+L_{l+1})(n-l)} = p(A, L_{l+1}; l). \end{aligned}$$

③$(n-l-1)L_{l+1} < A \le (n-l+1)L_{l+1} - (n-l-2)\Delta$.

$$p(A', L'_{l+1}; l) = \frac{4L'_{l+1}}{(A'+(n-l+1)L'_{l+1})^2} = \frac{4(L_{l+1}-\Delta)}{(A+(n-l+1)L_{l+1}-(n-l)\Delta)^2}.$$

$$\begin{aligned} &(A+(n-l+1)L_{l+1}-(n-l)\Delta)^2 L_{l+1} \\ =&(A+(n-l+1)L_{l+1})^2 L_{l+1} + (n-l)^2\Delta^2 L_{l+1} - 2\Delta(A+(n-l+1)L_{l+1})(n-l)L_{l+1} \\ &+\Delta(A+(n-l+1)L_{l+1})^2 + (n-l)^2 L_{l+1}^2\Delta - \Delta(A+(n-l+1)L_{l+1})^2 - (n-l)^2 L_{l+1}^2\Delta \\ =&\Delta(A+L_{l+1})^2 - (n-l)^2\Delta L_{l+1}(L_{l+1}-\Delta) + (A+(n-l+1)L_{l+1})^2(L_{l+1}-\Delta) \\ \ge&\Delta^2(n-l)^2 L_{l+1} + (A+(n-l+1)L_{l+1})^2(L_{l+1}-\Delta) \\ \ge&(A+(n-l+1)L_{l+1})^2(L_{l+1}-\Delta). \end{aligned}$$

Then

$$p(A', L'_{l+1}; l) \le \frac{4(L_{l+1}-\Delta)L_{l+1}}{(A+(n-l+1)L_{l+1})^2(L_{l+1}-\Delta)} = \frac{4L_{l+1}}{(A+(n-l+1)L_{l+1})^2} = p(A, L_{l+1}; l).$$

④$(n-l+1)L_{l+1} - (n-l-2)\Delta < A \le (n-l+1)L_{l+1}$. From the corresponding diagram of this case, we can observe $(n-l-1)L_{l+1} \le (n-l+1)L_{l+1} - (n-l-2)\Delta \Rightarrow (n-l-2)\Delta \le 2L_{l+1}$.

$$\begin{aligned} p(A', L'_{l+1}; l) &= \frac{1}{A'(n-l+1)} = \frac{1}{(A+\Delta)(n-l+1)} \\ &= \frac{1}{\Delta(n-l+1) + \frac{(A+(n-l+1)L_{l+1})^2}{4L_{l+1}} - \frac{(A-(n-l+1)L_{l+1})^2}{4L_{l+1}}}. \end{aligned}$$

According to the range of A in this case, we have

$$\frac{(A-(n-l+1)L_{l+1})^2}{4L_{l+1}} < \frac{(n-l-2)^2\Delta^2}{4L_{l+1}} \le \frac{(n-l-2)\Delta}{2},$$

$$\Rightarrow \Delta(n-l+1) - \frac{(A-(n-l+1)L_{l+1})^2}{4L_{l+1}} > \Delta(n-l+1) - \frac{(n-l-2)\Delta}{2} = \frac{\Delta}{2}(n-l+4) \ge 0.$$

$$p(A', L'_{l+1}; l) = \frac{1}{\Delta(n-l+1) + \frac{(A+(n-l+1)L_{l+1})^2}{4L_{l+1}} - \frac{(A-(n-l+1)L_{l+1})^2}{4L_{l+1}}}$$
$$< \frac{1}{\frac{(A+(n-l+1)L_{l+1})^2}{4L_{l+1}}} = \frac{4L_{l+1}}{(A+(n-l+1)L_{l+1})^2} = p(A, L_{l+1}; l).$$

⑤$A > (n-l+1)L_{l+1}$.

$$p(A', L'_{l+1}; l) = \frac{1}{A'(n-l+1)} < \frac{1}{A(n-l+1)} = p(A, L_{l+1}; l).$$

⑥$(n-l+1)L_{l+1} - (n-l-2)\Delta < A \le (n-l-1)L_{l+1}$.

$$\begin{aligned} p(A', L'_{l+1}; l) &= \frac{1}{A'(n-l+1)} = \frac{1}{(A+\Delta)(n-l+1)} \\ &= \frac{1}{(A+L_{l+1})(n-l) + A + L_{l+1} - (n-l+1)(L_{l+1}-\Delta)} \\ &< \frac{1}{(A+L_{l+1})(n-l) + (n-l+2)L_{l+1} - (n-l-2)\Delta - (n-l+1)(L_{l+1}-\Delta)} \\ &= \frac{1}{(A+L_{l+1})(n-l) + L_{l+1} + 3\Delta} \\ &< \frac{1}{(A+L_{l+1})(n-l)} = p(A, L_{l+1}; l). \end{aligned}$$

⑦$(n-l-1)L_{l+1} < A \le (n-l+1)L_{l+1}$. From the corresponding diagram of this case, we can observe $(n-l+1)L_{l+1} - (n-l-2)\Delta < (n-l-1)L_{l+1} \Rightarrow 2L_{l+1} < (n-l-2)\Delta$.

$$\begin{aligned} p(A', L'_{l+1}; l) &= \frac{1}{A'(n-l+1)} = \frac{1}{(A+\Delta)(n-l+1)} \\ &= \frac{1}{\Delta(n-l+1) + \frac{(A+(n-l+1)L_{l+1})^2}{4L_{l+1}} - \frac{(A-(n-l+1)L_{l+1})^2}{4L_{l+1}}}. \end{aligned}$$

According to the range of A in this case, we have

$$\frac{(A-(n-l+1)L_{l+1})^2}{4L_{l+1}} < \frac{(2L_{l+1})^2}{4L_{l+1}} = L_{l+1} < \frac{(n-l-2)\Delta}{2},$$

$$\Rightarrow \Delta(n-l+1) - \frac{(A-(n-l+1)L_{l+1})^2}{4L_{l+1}} > \Delta(n-l+1) - \frac{(n-l-2)\Delta}{2} = \frac{\Delta}{2}(n-l+4) \ge 0.$$

$$p(A', L'_{l+1}; l) = \frac{1}{\Delta(n-l+1) + \frac{(A+(n-l+1)L_{l+1})^2}{4L_{l+1}} - \frac{(A-(n-l+1)L_{l+1})^2}{4L_{l+1}}}$$
$$< \frac{1}{\frac{(A+(n-l+1)L_{l+1})^2}{4L_{l+1}}} = \frac{4L_{l+1}}{(A+(n-l+1)L_{l+1})^2} = p(A, L_{l+1}; l).$$

Thus, in each of the cases, we have $p(A', L'_{l+1}; l) \le p(A, L_{l+1}; l)$, and, actually, in four of the cases, $p(A', L'_{l+1}; l) < p(A, L_{l+1}; l)$, and the proof of Claim 16 is complete. ■

Claim 17 *Let l^* denote an optimal value of l. If $S_1 \neq \emptyset$, $l^* = \max\{S_1\}$, otherwise $l^* = \min\{S_2\}$.*

Proof. By Claim 16, $L_1 \geq L_2 \geq \cdots \geq L_n$, and recall that $S_0 = \{l \in \mathbb{N}^+ : \sum_{i=1}^{l} L_i \leq (n-l-1)L_{l+1}\}, S_1 = \{l \in \mathbb{N}^+ : (n-l-1)L_{l+1} < \sum_{i=1}^{l} L_i \leq (n-l+1)L_{l+1}\}, S_2 = \{l \in \mathbb{N}^+ : \sum_{i=1}^{l} L_i > (n-l+1)L_{l+1}\}$. Thus, the values of the boundary points defining S_0, S_1, S_2, i.e., $(n-l-1)L_{l+1}, (n-l+1)L_{l+1}$, decrease as l increases, while $\sum_{i=1}^{l} L_i$ increases as l increases. We conclude that if $l \in S_0$, then $l+1$ could belong to S_0, S_1 or S_2; if $l \in S_1$, then $l+1$ could belong to S_1 or S_2, and if $l \in S_2$, then $l+1$ must belong to S_2. Furthermore, for any $l_0 \in S_0, l_1 \in S_1, l_2 \in S_2$, we have $l_0 < l_1 < l_2$.

To prove Claim 17, we consider the following cases:

Case 1: If $l \in S_0$, then we show that $p(\sum_{i=1}^{l} L_i, L_{l+1}; l) \geq p(\sum_{i=1}^{l+1} L_i, L_{l+2}; l+1)$, regardless of which set $l+1$ belongs to.

Case 2: If $l \in S_1$, then we show that $p(\sum_{i=1}^{l} L_i, L_{l+1}; l) > p(\sum_{i=1}^{l+1} L_i, L_{l+2}; l+1)$ when $l+1 \in S_1$, and $p(\sum_{i=1}^{l} L_i, L_{l+1}; l) \leq p(\sum_{i=1}^{l+1} L_i, L_{l+2}; l+1)$ when $l+1 \in S_2$.

Case 3: If $l \in S_2$, then $l+1 \in S_2$, and we show that $p(\sum_{i=1}^{l} L_i, L_{l+1}; l) < p(\sum_{i=1}^{l+1} L_i, L_{l+2}; l+1)$.

Clearly, if we prove the assertions in the above three cases, then $l^* = \max\{S_1\}$, if S_1 exists, and $l^* = \min\{S_2\}$, otherwise.

Case 1: $l \in S_0$.

If $l+1 \in S_0$, we have $A + L_{l+1} \leq (n-l-2)L_{l+2}$,

$$\begin{aligned} p(A+L_{l+1}, L_{l+2}; l+1) &= \frac{1}{(n-l-1)(A+L_{l+1}+L_{l+2})} \\ &= \frac{1}{(n-l)(A+L_{l+1}) + (n-l-1)L_{l+2} - A - L_{l+1}} \\ &< \frac{1}{(n-l)(A+L_{l+1})} = p(A, L_{l+1}; l); \end{aligned}$$

if $l+1 \in S_1$,

$$\begin{aligned} p(A+L_{l+1}, L_{l+2}; l+1) &= \frac{4L_{l+2}}{(A+L_{l+1}+(n-l)L_{l+2})^2} \\ &\leq \frac{4L_{l+2}}{4(A+L_{l+1})(n-l)L_{l+2}} = \frac{1}{(n-l)(A+L_{l+1})} = p(A, L_{l+1}; l); \end{aligned}$$

if $l+1 \in S_2$,

$$p(A+L_{l+1}, L_{l+2}; l+1) = \frac{1}{(n-l)(A+L_{l+1})} = p(A, L_{l+1}; l).$$

Case 2: $l \in S_1$. Then, we have $(n-l)L_{l+1} < A + L_{l+1} \leq (n-l+2)L_{l+1}$.

If $l+1 \in S_1$,

$$p(A+L_{l+1}, L_{l+2}; l+1) = \frac{4L_{l+2}}{(A+L_{l+1}+(n-l)L_{l+2})^2} = \frac{4L_{l+2}L_{l+1}}{L_{l+1}(A+L_{l+1}+(n-l)L_{l+2})^2},$$

$$
\begin{aligned}
&L_{l+1}(A+L_{l+1}+(n-l)L_{l+2})^2 = L_{l+1}((A+L_{l+1})^2+2(n-l)L_{l+2}(A+L_{l+1})+(n-l)^2L_{l+2}^2) \\
=&L_{l+1}((A+L_{l+1})^2+(n-l)^2L_{l+1}^2)+2(n-l)L_{l+2}L_{l+1}(A+L_{l+1}) \\
&+L_{l+2}((A+L_{l+1})^2+(n-l)^2L_{l+1}^2)-L_{l+2}((A+L_{l+1})^2+(n-l)^2L_{l+1}^2) \\
=&L_{l+2}((A+L_{l+1})^2+(n-l)^2L_{l+1}^2)+2(n-l)L_{l+2}L_{l+1}(A+L_{l+1}) \\
&+(L_{l+1}-L_{l+2})((A+L_{l+1})^2-(n-l)^2L_{l+1}L_{l+2}) \\
>&L_{l+2}(A+(n-l+1)L_{l+1})^2+(L_{l+1}-L_{l+2})((n-l)^2L_{l+1}^2-(n-l)^2L_{l+1}L_{l+2}) \\
=&L_{l+2}(A+(n-l+1)L_{l+1})^2+(L_{l+1}-L_{l+2})^2(n-l)^2L_{l+1} \\
\geq&L_{l+2}(A+(n-l+1)L_{l+1})^2,
\end{aligned}
$$

$$
\begin{aligned}
p(A+L_{l+1},L_{l+2};l+1) &= \frac{4L_{l+2}L_{l+1}}{L_{l+1}(A+L_{l+1}+(n-l)L_{l+2})^2} \\
&< \frac{4L_{l+1}}{(A+(n-l+1)L_{l+1})^2} = p(A,L_{l+1};l);
\end{aligned}
$$

if $l+1\in S_2$,

$$
\begin{aligned}
p(A+L_{l+1},L_{l+2};l+1) &= \frac{1}{(n-l)(A+L_{l+1})} = \frac{4L_{l+1}}{4L_{l+1}(A+L_{l+1})(n-l)} \\
&\geq \frac{4L_{l+1}}{(A+L_{l+1}+(n-l)L_{l+1})^2} = p(A,L_{l+1};l).
\end{aligned}
$$

Case 3: $l\in S_2$. Then, $l+1\in S_2, A>(n-l+1)L_{l+1}$, and we have,

$$
\begin{aligned}
p(A+L_{l+1},L_{l+2};l+1) &= \frac{1}{(A+L_{l+1})(n-l)} = \frac{1}{A(n-l+1)+L_{l+1}(n-l)-A} \\
&> \frac{1}{A(n-l+1)} = p(A,L_{l+1};l).
\end{aligned}
$$

■